Complete Acyclic Colorings

GGTW 2019, Ghent, Belgium

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Arboreal and Acyclic Colorings

An arboreal coloring of a graph G is a partition of the vertex set into subsets inducing forests. It is complete if there is a cycle in the merge of any two color classes.

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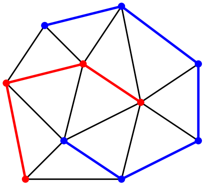
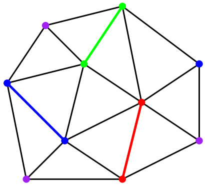
Vertex arboricity va(G)
Minimum number of colors in arboreal coloring



$\text{va}(G) = 2$

A-vertex arboricity ava(G)
Maximum number of colors in complete arboreal coloring



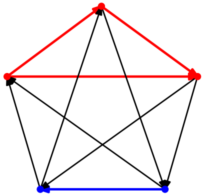
$\text{ava}(G) = 4$

Arboreal and Acyclic Colorings

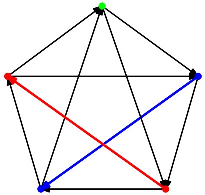
An acyclic coloring of a digraph D is a partition of the vertex set into subsets inducing acyclic digraphs. It is complete if there is a directed cycle in the merge of any two color classes.

Dichromatic number $\vec{\chi}(D)$
Minimum number of colors in acyclic coloring

Adichromatic number $\text{adi}(D)$
Maximum number of colors in complete acyclic coloring

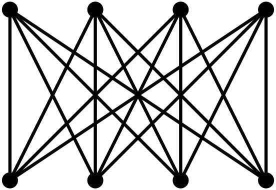


$\vec{\chi}(D) = 2$



$\text{adi}(D) = 3$

Complete Bipartite Graphs



$$\text{va}(K_{n,n}) = 2$$

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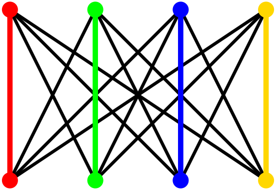
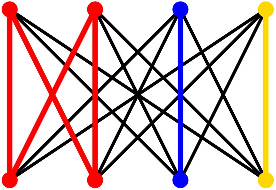


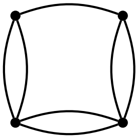
$$\text{ava}(K_{n,n}) = n$$

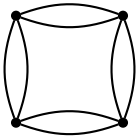
Complete Bipartite Graphs



$$\operatorname{ava}(K_{n,n}) = n$$

Subgraphs





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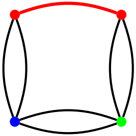
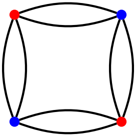


$\text{ava}(G') = 3$



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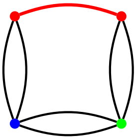

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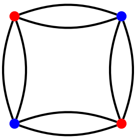

$\text{ava}(G) = 2$

Lemma

If G' is an induced subgraph of G, then $\text{ava}(G') \leq \text{ava}(G)$.

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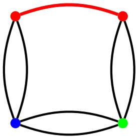

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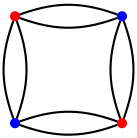

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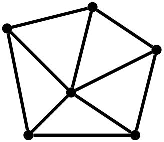

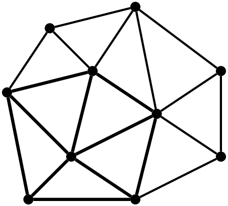

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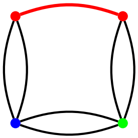

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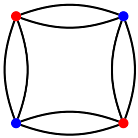

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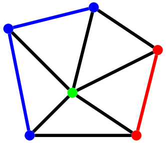

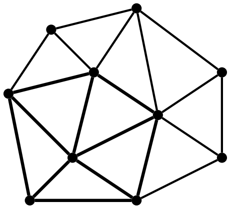

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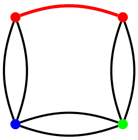

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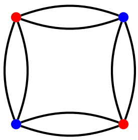

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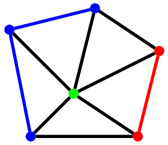

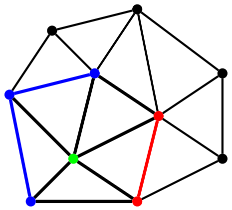

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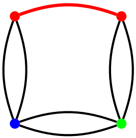

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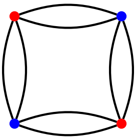

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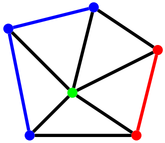

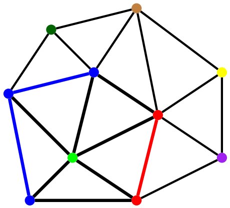

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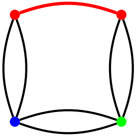

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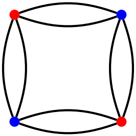

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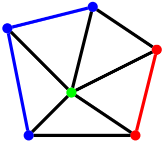
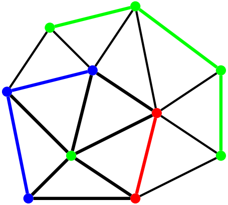
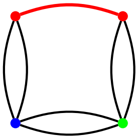




$\text{ava}(G) \geq 3$

Subgraphs



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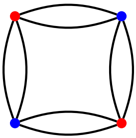

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Lemma

If G' is an induced subgraph of G, then $\mathrm{ava}(G') \leq \mathrm{ava}(G)$.

Lemma

If D' is an induced subdigraph of D, then $\mathrm{adi}(D') \leq \mathrm{adi}(D)$.

Induced Minors and Subdivisions

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If e is a simple edge, then $\operatorname{ava}(G/e) \leq \operatorname{ava}(G)$.

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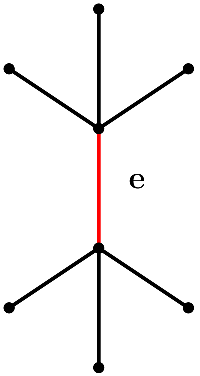

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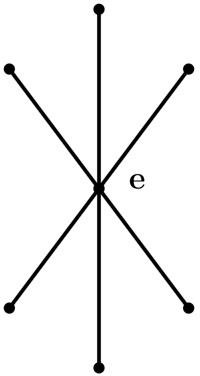

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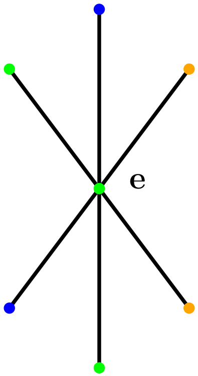

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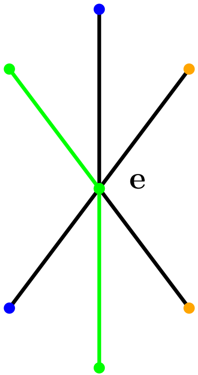

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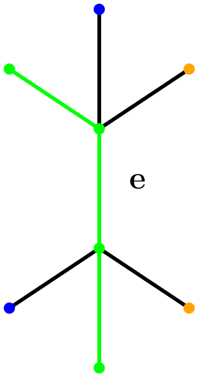
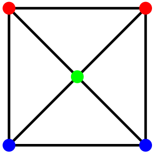


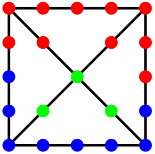
Induced Minors and Subdivisions

Corollary

If H is an induced minor of G, then $\text{ava}(H) \leq \text{ava}(G)$.



H



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Relation to Feedback Vertex Sets

Definition

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Proposition

- $\text{ava}(G) \leq \text{fv}(G) + 1$ for any graph G.
- $\text{adi}(D) \leq \text{fv}(D) + 1$ for any digraph D.

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Proposition

- $\text{ava}(G) \leq \text{fv}(G) + 1$ for any graph G.
- $\text{adi}(D) \leq \text{fv}(D) + 1$ for any digraph D.

Proof.

In a complete arboreal/acyclic coloring, at most one colour class is disjoint from a feedback vertex set. □

Relations between the Parameters

Theorem (Felsner, Hochstättler, Knauer, S. '19)

- $\exists$ *Multi-graphs with bounded* ava *and unbounded* fv.
- $\exists$ *Simple digraphs with bounded* adi *and unbounded* fv.
- *For simple graphs, there is f such that* $\mathrm{fv}(G) \leq f(\mathrm{ava}(G))$.
- *For simple graphs,* $\mathrm{ava}(G) \sim \max_D \mathrm{adi}(D)$.

Relations between the Parameters

Theorem (Felsner, Hochstättler, Knauer, S. '19)

Let $\mathcal{G}$ be a non-trivial minor-closed class of simple graphs.

- *There is f such that for D orientation of $G \in \mathcal{G}$:*

$$\mathrm{fv}(D) \leq f(\mathrm{adi}(D)).$$

- *There is $f(k) = O(k^2 \log k)$ such that for all $G \in \mathcal{G}$:*

$$\mathrm{fv}(G) \leq f(\mathrm{ava}(G)).$$

Degeneracy vs. ava

Theorem

There is f such that for all simple graphs G:

$$\deg(G) \leq f(\mathrm{ava}(G)).$$

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Theorem (Kühn and Osthus, 2004)

For $s \geq 1$ and every graph H there is $d(s,H) \geq 1$ such that every G with $\delta(G) \geq d(s,H)$ contains $K_{s,s}$ as a subgraph or an induced subdivision of H.

Degeneracy vs. ava

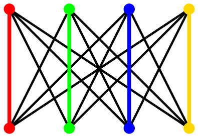
Theorem

There is f such that for all simple graphs G:

$$\deg(G) \leq f(\text{ava}(G)).$$

Proof.

If $\deg(G) \geq d(s, K_{s,s})$, then $\text{ava}(G) \geq s$. □



$\mathrm{fv}(G) \leq f(\mathrm{ava}(G))$

Proof by contradiction: Assume $\exists$ sequence $G_1, G_2, G_3, \ldots$ such that $\mathrm{fv}(G_i) \to \infty$ and $\mathrm{ava}(G_i)$ bounded.

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Theorem (Erdős and Pósa '65)

There is $f(k) = \mathcal{O}(k \log k)$ such that for all graphs:

$$\mathrm{cp}(G) \leq \mathrm{fv}(G) \leq f(\mathrm{cp}(G)).$$

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There is $f(k) = \mathcal{O}(k \log k)$ such that for all graphs:

$$\mathrm{cp}(G) \leq \mathrm{fv}(G) \leq f(\mathrm{cp}(G)).$$

Therefore: $\mathrm{cp}(G_i) \to \infty$, and $\deg(G_i) \leq d$.

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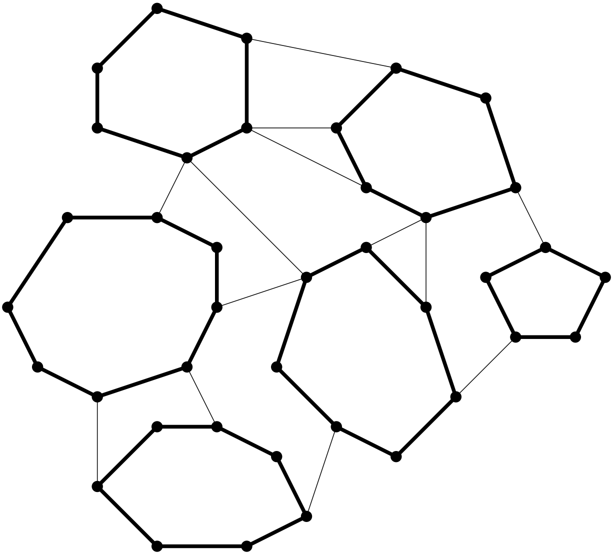

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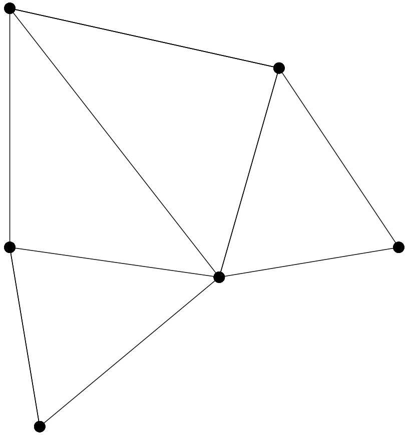

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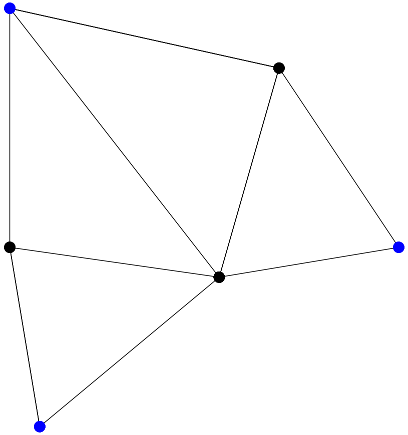

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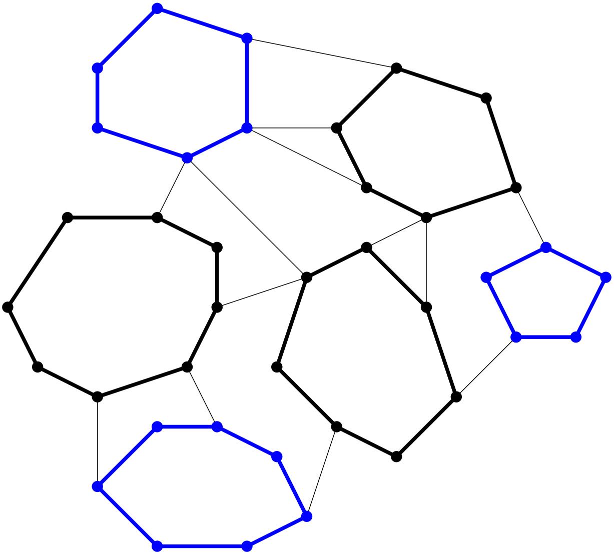

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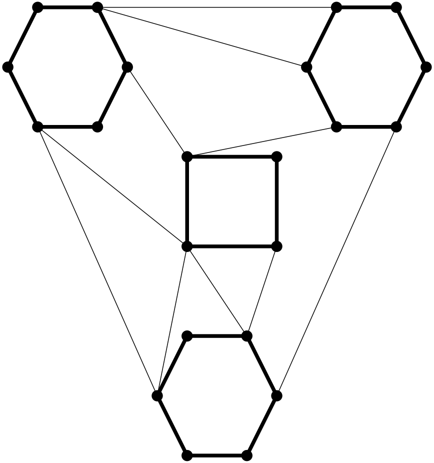

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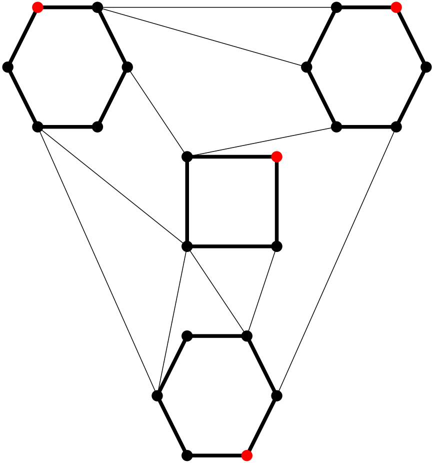

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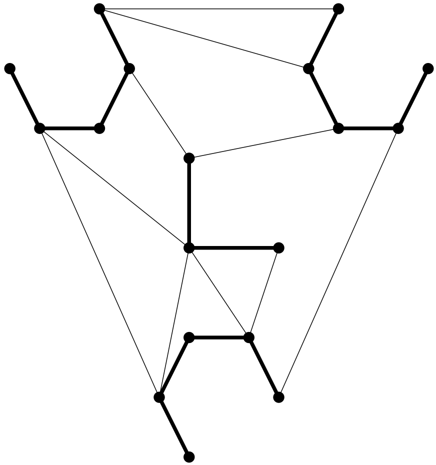

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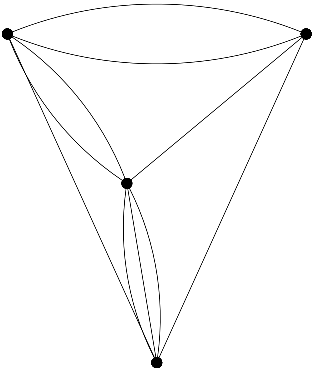

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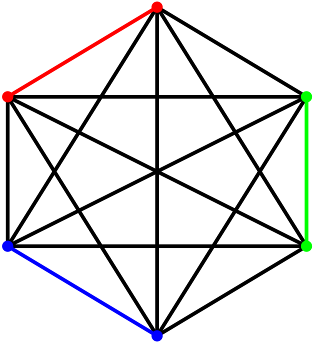
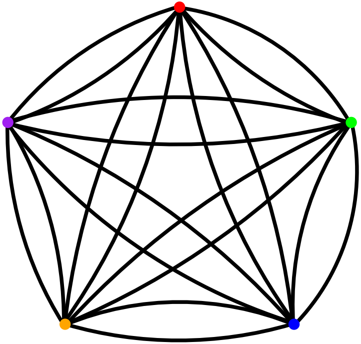




The End.

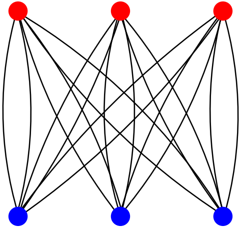
Thank you.

$\mathrm{fv}(G) \leq f(\mathrm{ava}(G))$

- k: Number of short cycles.
- L: Maximum length of short cycle.
- d: Upper bound for degeneracy.
- N: Number of vertices in short cycles.

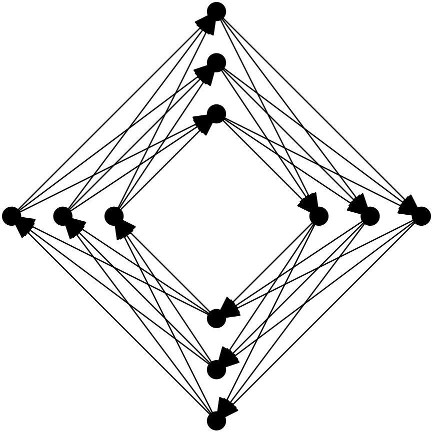
$$\binom{k}{2} \leq \text{Number of edges in induced subgraph} \leq dN \leq dL \cdot k$$

Multigraphs



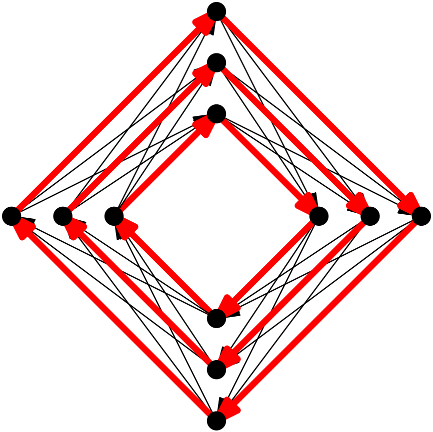
$\text{ava} = 2, \ \text{fv} = n$

Simple digraphs



$$n = 3, \ k = 4$$

Simple digraphs



$$\text{adi} \leq k, \;\; \text{fv} = n$$

Relationship of ava and adi

Relationship of ava and adi

$$\text{adi}(D) \leq \text{fv}(D) + 1 \leq \text{fv}(G) + 1 \leq f(\text{ava}(G)) + 1.$$

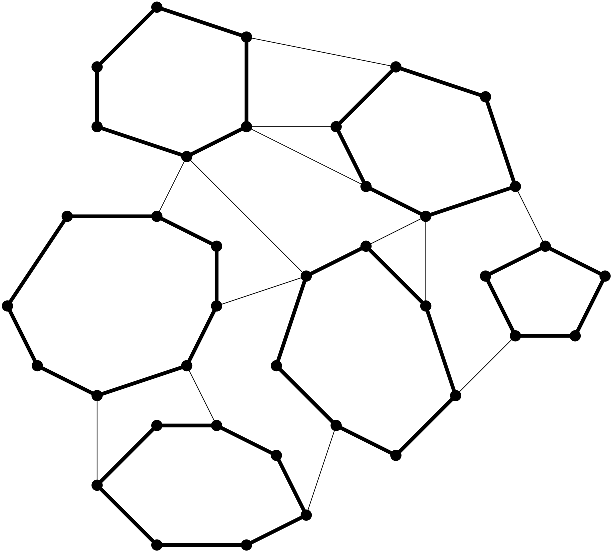
Relationship of ava and adi

$$\max_{D} \mathrm{adi}(D) \leq g(\mathrm{ava}(G)).$$

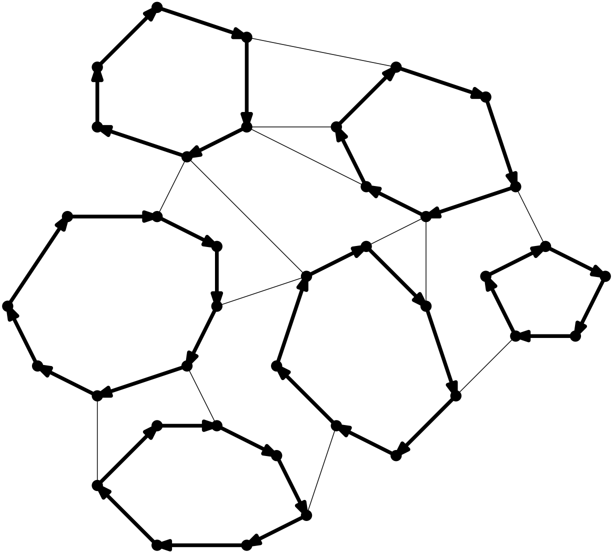
Relationship of ava and adi

$$\text{ava}(G) \leq h(\max_{D} \text{adi}(D)))?$$

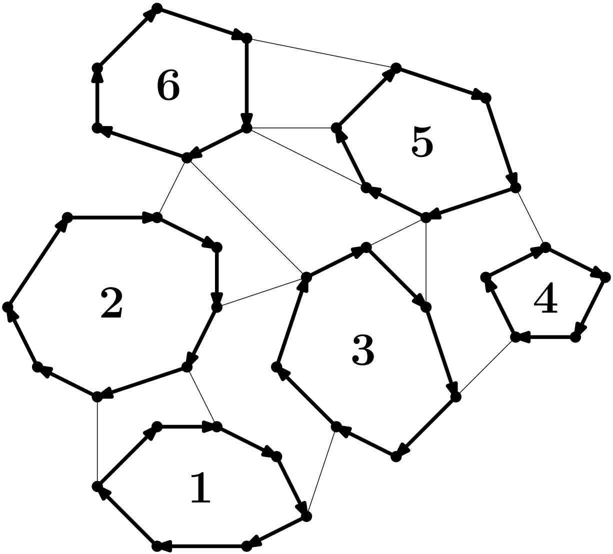
Relationship of ava and adi



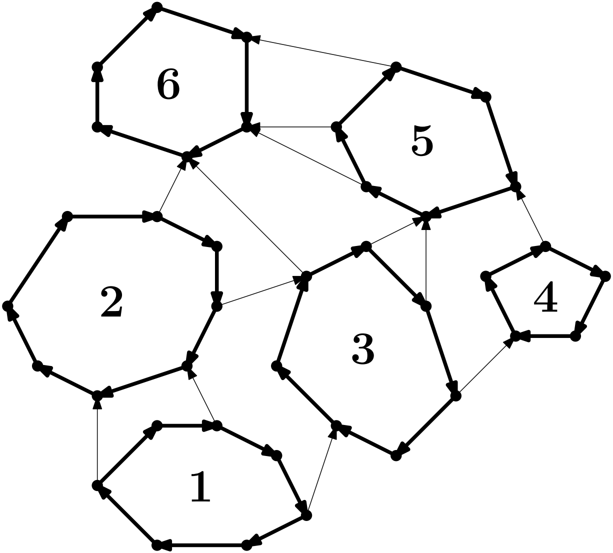
Relationship of ava and adi



Relationship of ava and adi



Relationship of ava and adi



Relationship of ava and adi