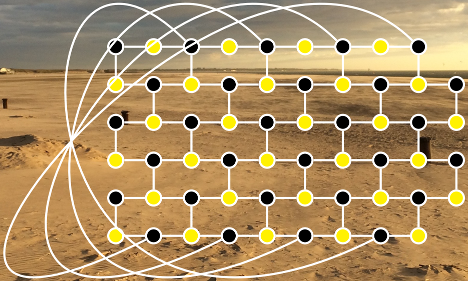
The Stable Set Problem in Graphs with Bounded Genus and Bounded Odd Cycle Packing Number



Samuel Fiorini (ULB) joint with
Michele Conforti, Tony Huyhn, Gwenaël Joret, and Stefan Weltge

Maximum Weight Stable Set

Problem

Given a graph G and $w : V(G) \to \mathbb{R}_{\geq 0}$, compute a maximum weight stable set ***(MWSS)*** *of G.*

Maximum Weight Stable Set

Problem

Given a graph G and $w : V(G) \to \mathbb{R}_{\geq 0}$, compute a maximum weight stable set ***(MWSS)*** *of G.*

Theorem

For every $\epsilon > 0$, it is NP*-hard to approximate maximum stable set within a factor of $n^{1-\epsilon}$.*

Bipartite Graphs

Theorem

MWSS *can be solved on bipartite graphs in polynomial time.*

Bipartite Graphs

Theorem

MWSS *can be solved on bipartite graphs in polynomial time.*

$$\begin{array}{rl} \max & \sum_{v \in V(G)} w(v) x_v \\ \text{s.t.} & x_u + x_v \leqslant 1 \quad \forall uv \in E(G) \\ & x \geqslant \mathbf{0} \end{array} \quad = \quad \begin{array}{rl} \max & \sum_{v \in V(G)} w(v) x_v \\ \text{s.t.} & Mx \leqslant \mathbf{1} \\ & x \geqslant \mathbf{0} \end{array}$$

Bipartite Graphs

Theorem

MWSS *can be solved on bipartite graphs in polynomial time.*

$$\begin{array}{ll} \max & \sum_{v \in V(G)} w(v) x_v \\ \text{s.t.} & x_u + x_v \leqslant 1 \quad \forall uv \in E(G) \\ & x \geqslant \mathbf{0} \end{array} \quad = \quad \begin{array}{ll} \max & \sum_{v \in V(G)} w(v) x_v \\ \text{s.t.} & Mx \leqslant \mathbf{1} \\ & x \geqslant \mathbf{0} \end{array}$$

If G is **bipartite**, then M is a **totally unimodular** matrix.

Integer Programming

Conjecture

Fix $k \in \mathbb{N}$. Integer Linear Programming can be solved in polynomial time when all subdeterminants of the constraint matrix are in $\{-k, \ldots, k\}$.

Integer Programming

Conjecture

Fix $k \in \mathbb{N}$. Integer Linear Programming can be solved in polynomial time when all subdeterminants of the constraint matrix are in $\{-k, \ldots, k\}$.

Theorem (Artmann, Weismantel, Zenklusen '17)

True for $k = 2$. Bimodular Integer Programming can be solved in (strongly) polynomial time.

Integer Programming

Conjecture

Fix $k \in \mathbb{N}$. Integer Linear Programming can be solved in polynomial time when all subdeterminants of the constraint matrix are in $\{-k, \ldots, k\}$.

Theorem (Artmann, Weismantel, Zenklusen '17)

True for $k = 2$. Bimodular Integer Programming can be solved in (strongly) polynomial time.

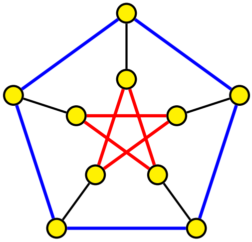
Open for $k \geq 3$.

Odd Cycle Packing Number

$M = M(G)$ *edge-vertex incidence matrix* of graph G

Odd Cycle Packing Number

$M = M(G)$ *edge-vertex incidence matrix* of graph G



$$M = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Odd Cycle Packing Number

Observation

$$\max |\textit{sub-determinant of } M(G)| = 2^{\mathsf{OCP}(G)}$$

Odd Cycle Packing Number

Observation

$$\max |\textit{sub-determinant of } M(G)| = 2^{\mathsf{OCP}(G)}$$

Corollary

MWSS *can be solved in polynomial time in graphs without two vertex-disjoint odd cycles.*

Odd Cycle Packing Number

Observation

$$\max |\textit{sub-determinant of } M(G)| = 2^{\mathsf{OCP}(G)}$$

Corollary

***MWSS** can be solved in polynomial time in graphs without two vertex-disjoint odd cycles.*

Conjecture

Fix $k \in \mathbb{N}$. **MWSS** can be solved in polynomial time in graphs without k vertex-disjoint odd cycles.

Polynomial Time Approximation Schemes

Theorem (Bock, Faenza, Moldenhauer, Ruiz-Vargas '14)

For every ***fixed*** $k \in \mathbb{N}$*,* ***MWSS*** *on graphs with* $\mathrm{OCP}(G) \leqslant k$ *has a PTAS.*

Polynomial Time Approximation Schemes

Theorem (Bock, Faenza, Moldenhauer, Ruiz-Vargas '14)

*For every **fixed** $k \in \mathbb{N}$, **MWSS** on graphs with $\mathrm{OCP}(G) \leqslant k$ has a PTAS.*

Theorem (Tazari '10)

*For every **fixed** $k \in \mathbb{N}$, **MWSS** and Minimum Vertex Cover on graphs with $\mathrm{OCP}(G) \leqslant k$ has a PTAS.*

Our Main Result

Theorem (CFHJW '19)

There exists a function $f : \mathbb{N}^2 \to \mathbb{N}$ such that ***MWSS*** *on graphs with $\mathrm{OCP}(G) \leqslant k$ and Euler genus $\leqslant g$ can be solved in $n^{O(f(k,g))} + n^{O(g^2)}$ time.*

Escher Walls

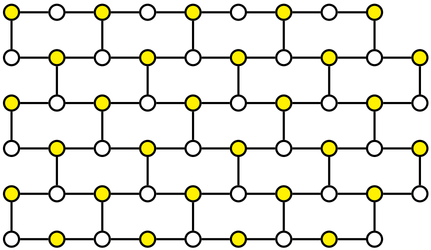
Observation

If $\exists$ **small** $X \subseteq V(G)$ such that $G - X$ is **bipartite**, then **MWSS** is easy

Escher Walls

Observation

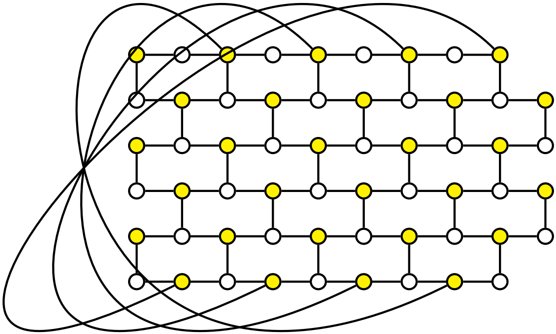
If $\exists$ **small** $X \subseteq V(G)$ such that $G - X$ is **bipartite**, then **MWSS** is easy



Escher Walls

Observation

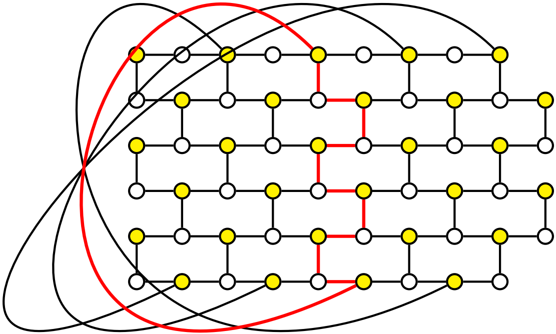
If $\exists$ **small** $X \subseteq V(G)$ such that $G - X$ is **bipartite**, then **MWSS** is easy



Escher Walls

Observation

If $\exists$ **small** $X \subseteq V(G)$ such that $G - X$ is **bipartite**, then **MWSS** is easy



$\mathrm{OCP} = 1$ **Graphs**

Theorem (Lovász)

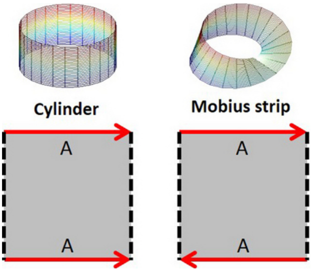
Let G be an internally 4-connected graph. Then $\mathrm{OCP}(G) \leqslant 1$ iff one of the following holds:

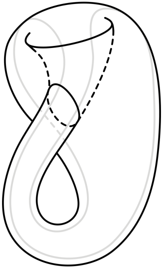
- $|G| \leqslant 5$,
- $G - \{x\}$ *is bipartite for some* $x \in V(G)$,
- $G - \{e_1, e_2, e_3\}$ *is bipartite for some* 3*-cycle* $\{e_1, e_2, e_3\} \subseteq E(G)$,
- G *has an even face embedding in the projective plane.*

Parity-consistent Embeddings

Definition

Let G be a graph embedded in a surface $\mathbb{S}$. A cycle of G is 1-*sided* if it has a neighborhood that is a **Möbius strip**, and 2-*sided* if it has a neighborhood that is a **cylinder**.





Parity-consistent Embeddings

Definition

The embedding of a graph G in a surface $\mathbb{S}$ is **parity-consistent** if every odd cycle in G is 1-sided.

Parity-consistent Embeddings

Definition

The embedding of a graph G in a surface $\mathbb{S}$ is **parity-consistent** if every odd cycle in G is 1-sided.

Lemma

If $G \hookrightarrow \mathbb{S}$ is parity-consistent, then $\mathrm{OCP}(G) \leqslant$ Euler genus$(\mathbb{S})$.

An Erdős-Pósa Theorem for 2-sided Odd Cycles

Theorem (CFHJW '19)

*Let $\mathbb{S}$ be a surface with Euler genus g. $\forall$ $\mathrm{OCP} \leqslant k$ graphs G embedded in $\mathbb{S}$, $\exists$ set X of $f(k, g)$ nodes that **hits all the 2-sided odd cycles**.*

An Erdős-Pósa Theorem for 2-sided Odd Cycles

Theorem (CFHJW '19)

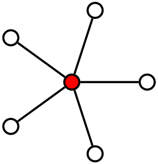
*Let $\mathbb{S}$ be a surface with Euler genus g. $\forall$ $\mathrm{OCP} \leqslant k$ graphs G embedded in $\mathbb{S}$, $\exists$ set X of $f(k, g)$ nodes that **hits all the 2-sided odd cycles**.*

Theorem (Reed '99, Kawarabayashi and Nakamoto '07)

*$\forall$ $\mathrm{OCP} \leqslant k$ graphs G embedded in an **orientable** surface $\mathbb{S}$ with Euler genus g, $\exists$ set X of $\tilde{f}(k, g)$ nodes that $G - X$ is bipartite.*

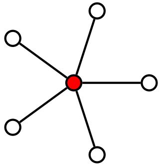
Slack Space

- **Node space:** $$x_v = \begin{cases} 1 & \text{if } v \in S \\ 0 & \text{otherwise} \end{cases}$$

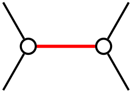


Slack Space

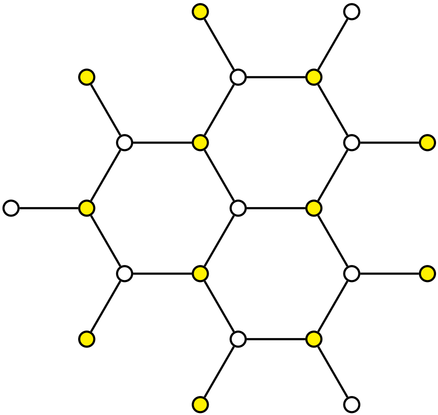
- **Node space:** $x_v = \begin{cases} 1 & \text{if } v \in S \\ 0 & \text{otherwise} \end{cases}$



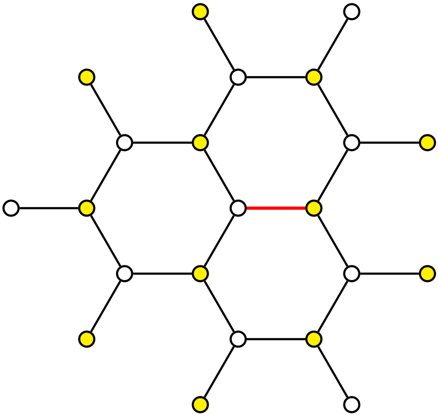
- **Slack space:** $y_{uv} = \begin{cases} 1 & \text{if } u, v \notin S \\ 0 & \text{otherwise} \end{cases}$



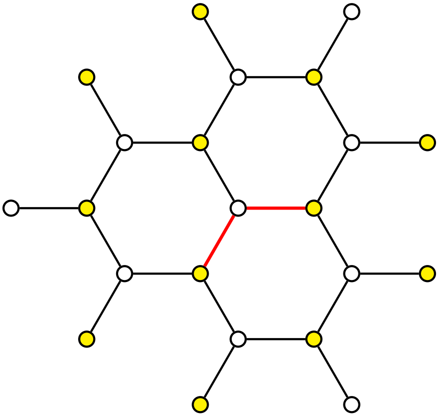
Slack Space



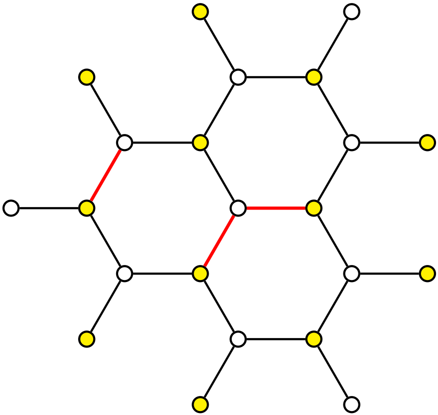
Slack Space



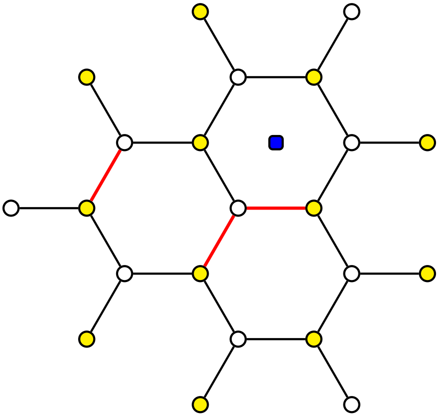
Slack Space



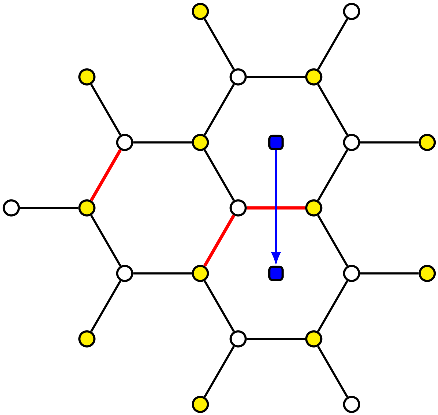
Slack Space



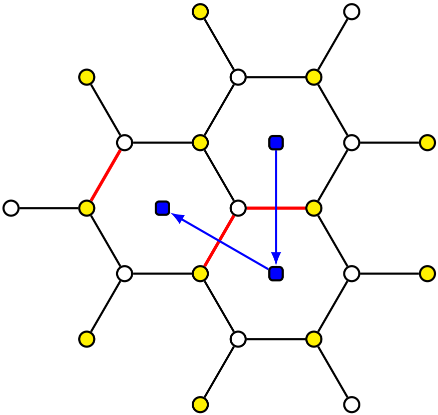
Slack Space



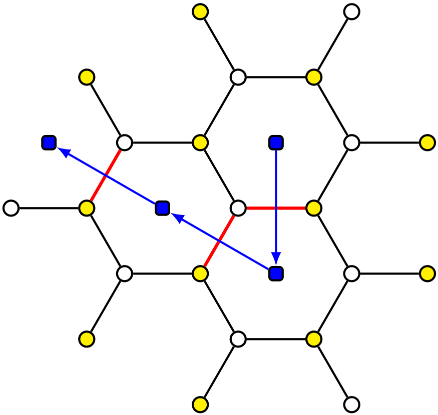
Slack Space



Slack Space



Slack Space



Minimum Cost Homologous Circulation

$$\begin{aligned} \max \quad & \sum_{v \in V(G)} w(v) x_v \\ \text{s.t.} \quad & Mx \leqslant \mathbf{1} \\ & x \geqslant \mathbf{0} \\ & x \in \mathbb{Z}^{V(G)} \end{aligned} \qquad \equiv \qquad \begin{aligned} \min \quad & \sum_{e \in E(G)} c(e) y_e \\ \text{s.t.} \quad & y \text{ circulation in } G^* \\ & y \text{ homologous to } \mathbf{1} \\ & y \geqslant \mathbf{0} \\ & y \in \mathbb{Z}^{E(G)} \end{aligned}$$

Minimum Cost Homologous Circulation

$$\begin{array}{rl} \max & \sum_{v \in V(G)} w(v) x_v \\ \text{s.t.} & Mx \leqslant \mathbf{1} \\ & x \geqslant \mathbf{0} \\ & x \in \mathbb{Z}^{V(G)} \end{array} \equiv \begin{array}{rl} \min & \sum_{e \in E(G)} c(e) y_e \\ \text{s.t.} & y \text{ circulation in } G^* \\ & y \text{ homologous to } \mathbf{1} \\ & y \geqslant \mathbf{0} \\ & y \in \mathbb{Z}^{E(G)} \end{array}$$

where $c \in \mathbb{R}_+^{E(G)}$ is such that $c(\delta(v)) = w(v)$ for all $v \in V(G)$

Homology

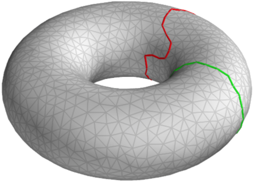
Definition

Two integer circulations y, y' in G^* are *homologous* if $y - y'$ is an integer combination of **facial** circulations.

Homology

Definition

Two integer circulations y, y' in G^* are *homologous* if $y - y'$ is an integer combination of **facial** circulations.



Minimum Cost Homologous Circulation

Theorem (Chambers, Erickson, Nayyeri '10)

Given an undirected graph G embedded on an orientable surface of Euler genus g, a cost function $c : E(G) \to \mathbb{R}$, and a circulation $\theta : E(G) \to \mathbb{R}$, a min-cost circulation $\mathbb{R}$-homologous to θ can be computed in time $g^{O(g)} n^{3/2}$.

Minimum Cost Homologous Circulation

Theorem (Chambers, Erickson, Nayyeri '10)

Given an undirected graph G embedded on an orientable surface of Euler genus g, a cost function $c : E(G) \to \mathbb{R}$, and a circulation $\theta : E(G) \to \mathbb{R}$, a min-cost circulation $\mathbb{R}$-homologous to θ can be computed in time $g^{O(g)} n^{3/2}$.

Theorem (Malnič and Mohar '92)

Suppose G is embedded in a surface $\mathbb{S}$ with Euler genus $g \geqslant 1$. If C_1, ..., C_ℓ are vertex-disjoint directed cycles in G ***whose homology classes are mutually distinct****, then $\ell \leqslant 6g$.*

Summary

Theorem (CFHJW '19)

There exists a function $f : \mathbb{N}^2 \to \mathbb{N}$ such that ***MWSS*** *on graphs with $\mathrm{OCP}(G) \leqslant k$ and Euler genus $\leqslant g$ can be solved in $n^{O(f(k,g))} + n^{O(g^2)}$ time.*

Summary

Theorem (CFHJW '19)

There exists a function $f : \mathbb{N}^2 \to \mathbb{N}$ *such that* ***MWSS*** *on graphs with* $\mathrm{OCP}(G) \leqslant k$ *and Euler genus* $\leqslant g$ *can be solved in* $n^{O(f(k,g))} + n^{O(g^2)}$ *time.*

Thank you!