On the 4-color theorem for signed graphs

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Signed graphs : definition

A signed graph (G, σ) is a pair where G is the *underlying graph* and

$$\sigma : \quad E(G) \quad \longrightarrow \quad \{-1, 1\}$$

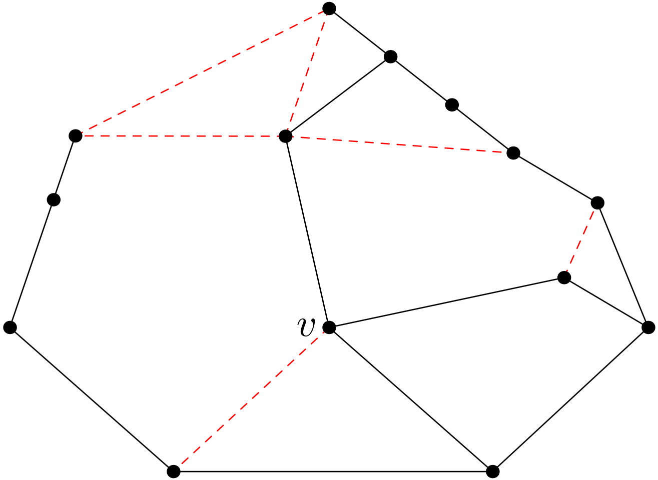
is called a *signature*.

Signed graphs : switching a vertex

Switching at a vertex v is switching the sign of the incident edges :
$s_v((G, \sigma)) = (G, \sigma')$ where :

$$\sigma'(e) = \begin{cases} -\sigma(e) & \text{if } e \text{ is incident to } v, \\ \sigma(e) & \text{otherwise.} \end{cases}$$

Signed graphs : switching a vertex



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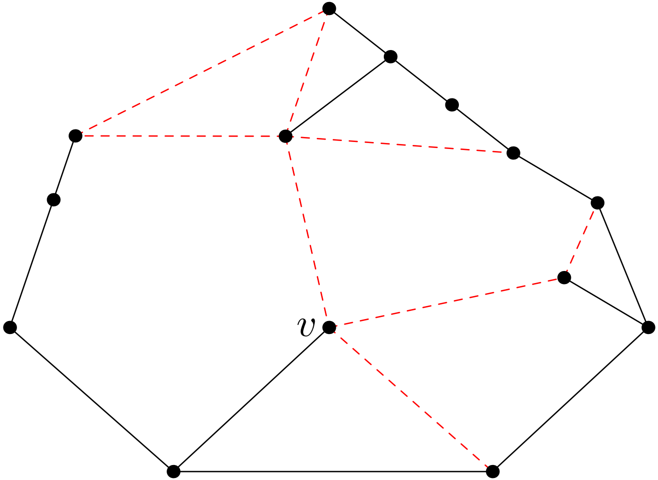

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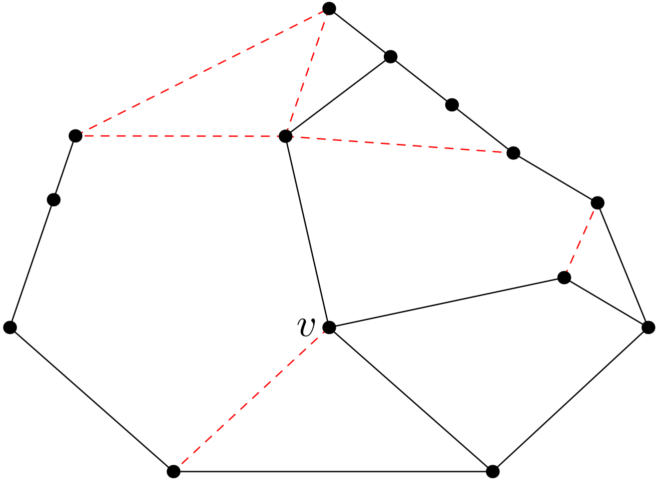
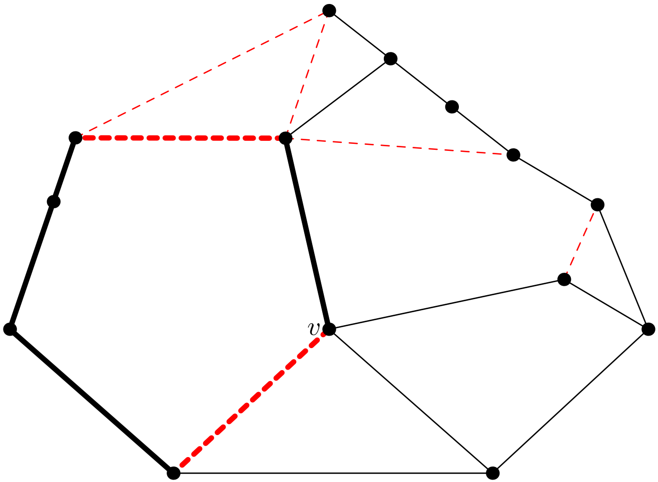


Signed graphs : sign of the cycles



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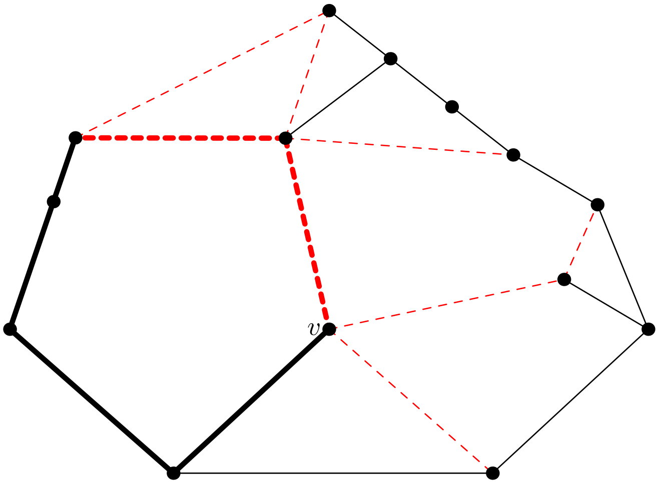


The sign of cycles is preserved when switching.

Signed graphs : cycle characterization

- Two signed graphs are switching-equivalent iff their cycles have the same sign.
- It suffices to consider a cycle base.
- For planar graphs we can consider the **facial cycles**.
- A cycle is **balanced** if it has an **even** number of negative edges.
- A cycle is **unbalanced** if it has an **odd** number of negative edges.

Signed graphs : coloring

Zaslavsky (1982) ; Máčajová, Raspaud and Škoviera (2016) :

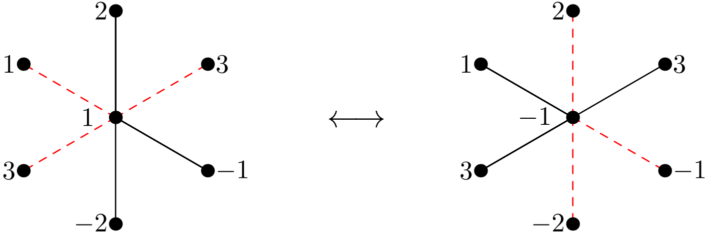
a signed k-coloring :

$$c: \; V(G) \longrightarrow \begin{array}{l} \{-k/2, ..., -1, 1, ..., k/2\} \text{ if } k \text{ is even} \\ \{-\lfloor k/2 \rfloor, ..., -1, 0, 1, ..., \lfloor k/2 \rfloor\} \text{ if } k \text{ is odd} \end{array}$$

s.t. $c(u) \neq \sigma(uv) \cdot c(v)$.

We denote $\chi(G)$ the minimum k s.t. such a coloring exists.

Signed graphs : coloring and switching



To preserve the coloring when switching at a vertex, it suffices to switch the sign of the color.

Extension of results of proper coloring

Theorem (Máčajová, Raspaud, Škoviera, 2016)

Let G be a simple connected signed graph. If G is not the ***balanced complete graph****, an* ***balanced odd cycle*** *or an* ***unbalanced even cycle****, then* $\chi(G) \leq \Delta$.

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- *If G is triangle-free, then $\chi(G) \leq 4$.*
- *If G has girth at least 5, then $\chi(G) \leq 3$.*

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Theorem (Jin, Kang, Steffen, 2016)

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Signed graphs : 4-CT for signed graphs ?

Conjecture (Máčajová, Raspaud, Škoviera, 2016)

Every signed planar graph is 4-signed-colorable.

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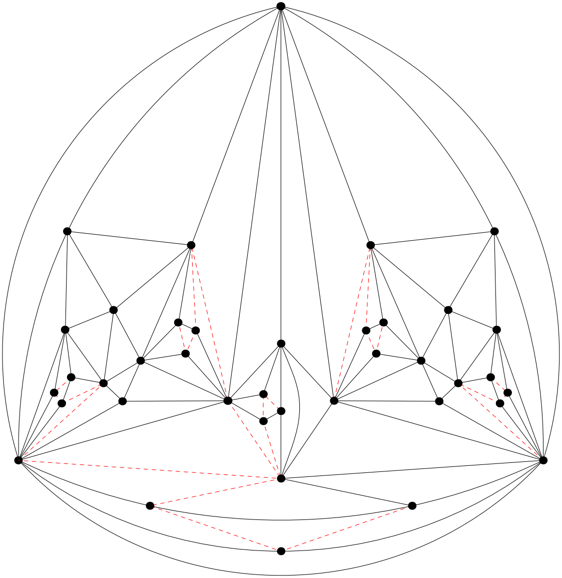
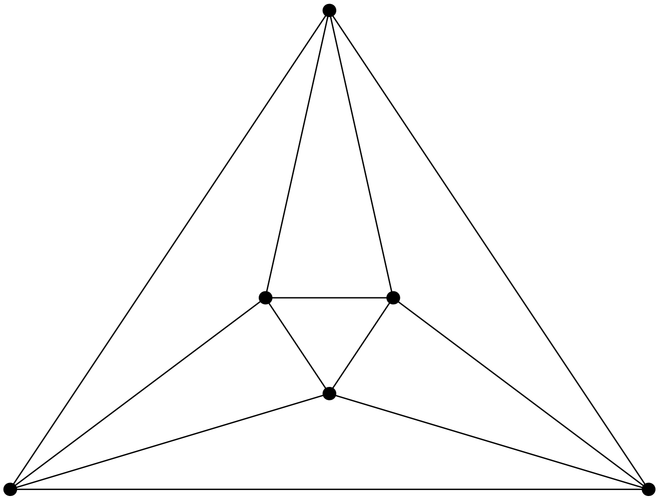
Theorem (K., Narboni, 2019+)

There exists a signed planar graph on 39 vertices that is not 4-signed-colorable.

A 39-vertex non-4-signed-colorable graph



Non signed triangulations : 4-coloring reduces to 3-edge-coloring of the dual



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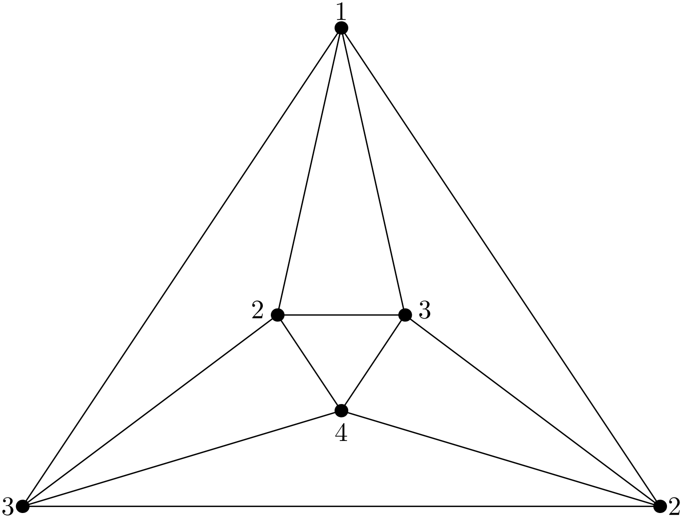

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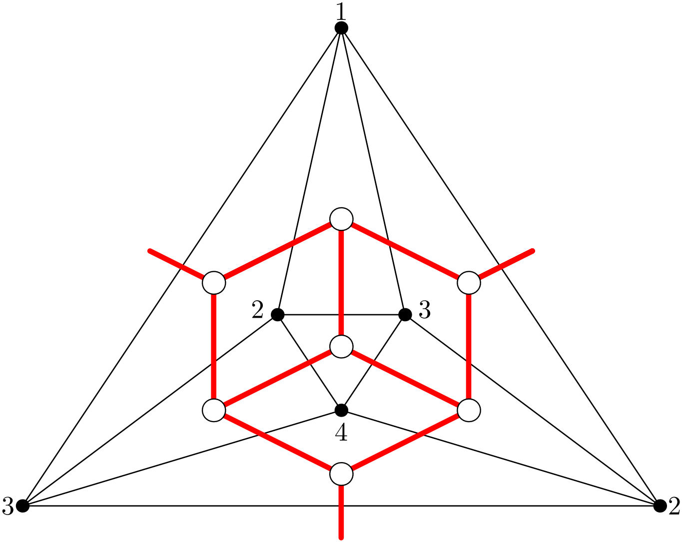
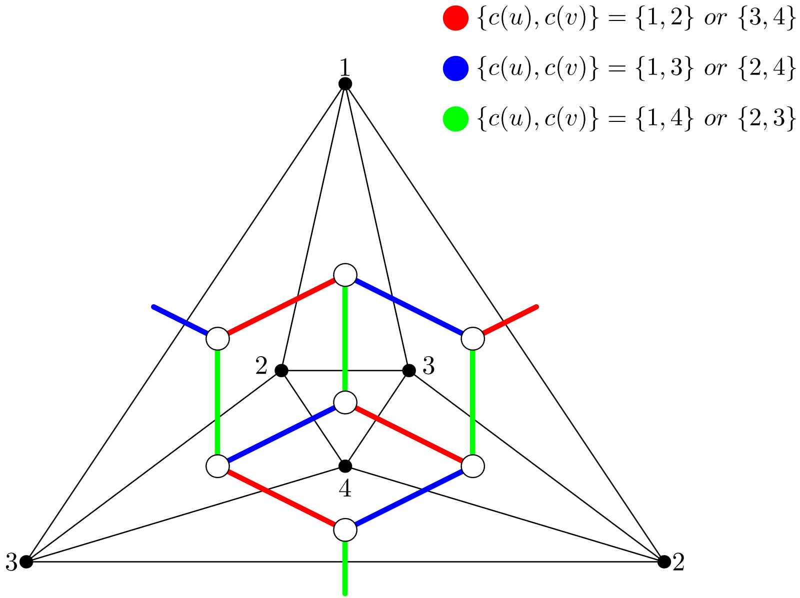
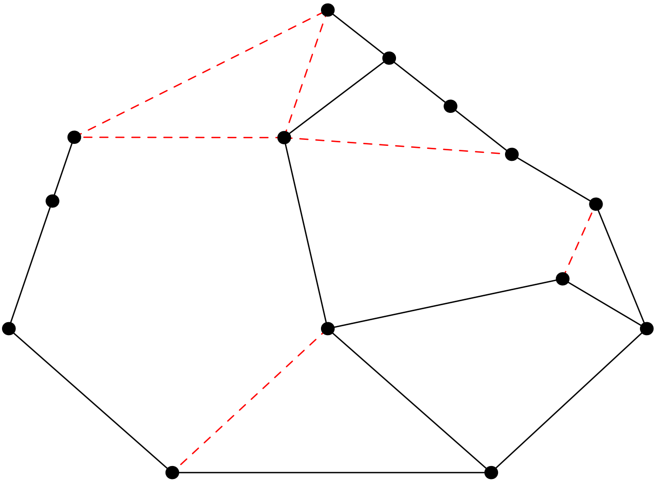


3-edge coloring of the dual



Dual graph of a signed planar graph



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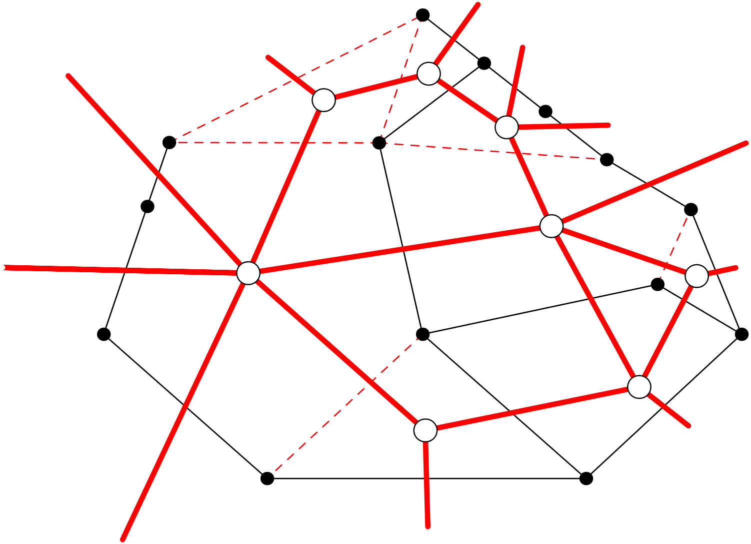

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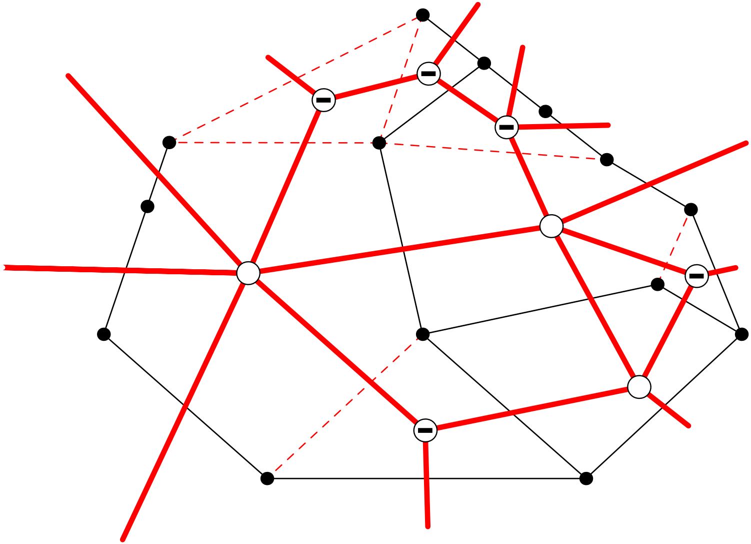

Switching preserves the signs of the vertices.

Dual graph of a signed planar graph : definition

Let (G, σ) be a 3-connected planar signed graph, the dual is :

(G^*, τ) where G^* is the dual of G, and

$$\tau : \quad V(G^*) \quad \longrightarrow \quad \{-1, 1\}$$

Where $\tau(f^*) = \sigma(f)$, with f^* being the vertex of G^* corresponding to the face f of G.

Dual graph of a signed planar graph : properties

- The vertices of G^* have a sign, this sign is preserved by switching.
- In G^*, there is always an even number of negative vertices.
- If G is a triangulation, G^* is cubic, so it also has an even number of positive vertices.

From 4-colorings to consistent 2-factors

Every planar graph is **4-signed-colorable**.

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Consistent 2-factor

Let G^* be a 3-connected cubic planar graph with an even number of negative vertices. A 2-factor of G^* is consistent if each cycle in the 2-factor is incident to an **even number of positive vertices**.

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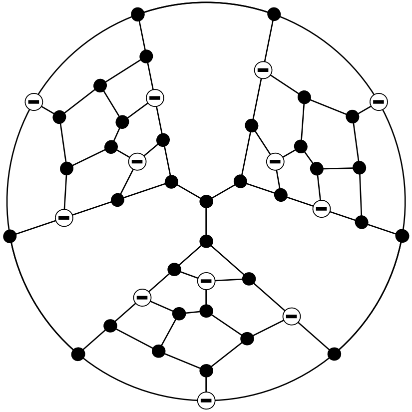
If G^* is **hamiltonian**, then it has a consistent 2-factor.

A cubic graph with no consistent 2-factor

Theorem

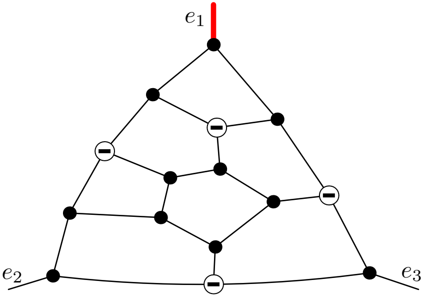
The Tutte's graph with a choice of negative vertices as depicted in the following figure does not admit a consistent 2-factor.



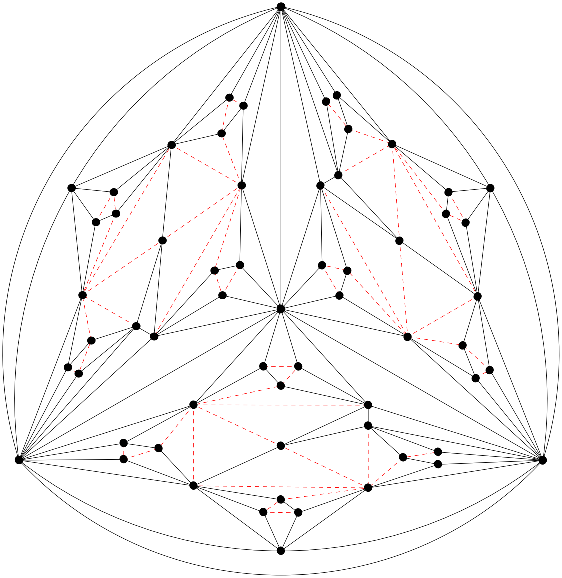
The Tutte's fragment

Lemma

Let G^* be a 3-connected cubic planar graph with an even number of negative vertices, containing a Tutte's fragment T_0 attached by the edges e_1, e_2, e_3, as depicted in the following figure. Then every consistent 2-factor of G^* contains the edge e_1.

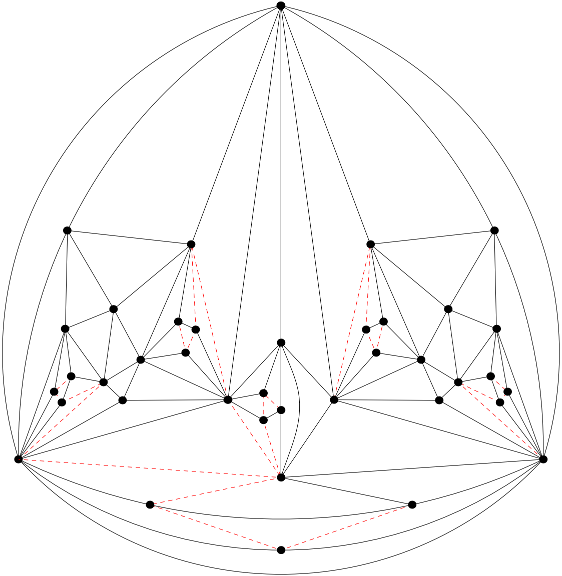


A graph with no 4-signed-coloring



Future work

- Search for a minimum counter-example ($21 \leq n \leq 39$).
- Study the complexity of deciding if a signed planar is 4-colorable.
- Try to translate other types of coloring to edge labeling of the dual (e.g., pair list coloring).



Thank you !