Superposition of snarks revisited

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joint work with Edita Máčajová

Cubic graphs

Every cubic graph can be properly coloured with four colours [Vizing 1964]
$\Longrightarrow$ cubic graphs naturally split into two classes:

Class 1 . . . graphs that admit a 3-edge-colouring ($\chi' = 3$)
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- almost all cubic graphs are Class 1 [Robinson & Wormald 1992]
- deciding whether a cubic graph is Class 1 or Class 2 is difficult [Holyer 1981]

- Class 2 graphs rare, difficult to understand ... and important

Snarks are 'nontrivial' cubic graphs of Class 2.

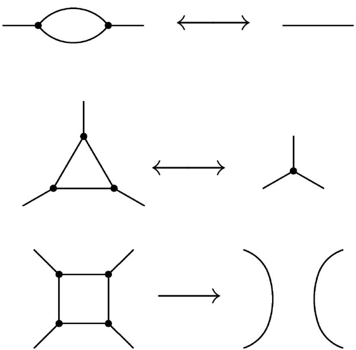
Snarks

Snarks are crucial for many important problems and conjectures in graph theory:

- Four-Colour-Theorem/Problem
- Cycle Double-Cover Conjecture
- 5-Flow Conjecture
- Fulkerson's Conjecture
- etc.

– trivially true for 3-edge-colourable graphs
– open for snarks
– potential counterexamples are usually snarks with very special properties

Nontrivial snarks



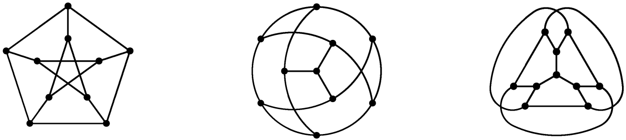
Similar simplifications for cycle-separating edge-cuts of size ≤ 3

$\Longrightarrow$ 'nontrivial' usually means

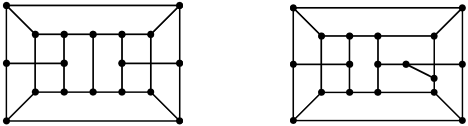
- girth > 4, and
- cyclically 4-edge-connected.

Early snarks

- Petersen graph [Kempe 1886; Petersen 1898]

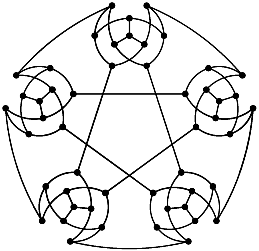


- Blanuša snarks of order 18
  [Blanuša 1946] [Adelson-Velskii & Titov 1973]



Early snarks

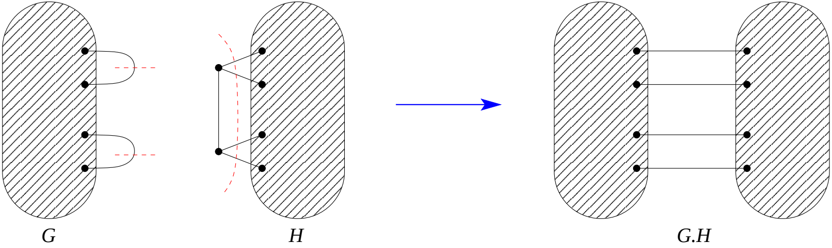
- Blanche Descartes snark of order 210 [Tutte 1948]
- Szekeres snark of order 50 [Szekeres 1973]



- infinitely many nontrivial snarks
  [Adelson-Velskii & Titov 1973; Isaacs 1975]

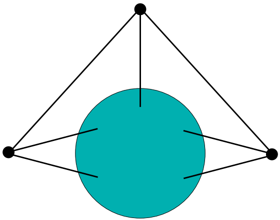
Dot product

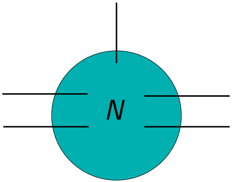
- Introduced in [Isaacs 1975] and [Adelson-Velskii & Titov 1973]



If G *and* H *are snarks, then* G.H *is a snark.*
If both G *and* H *are cyclically 4-edge-connected, then so is* G.H*.*

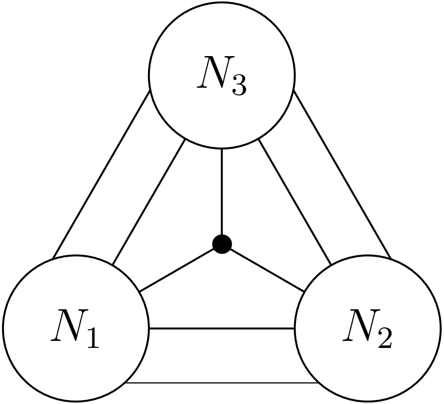
Negator construction





[Loupekine (Isaacs) 1976; Goldberg 1981]

Negator construction



Superposition

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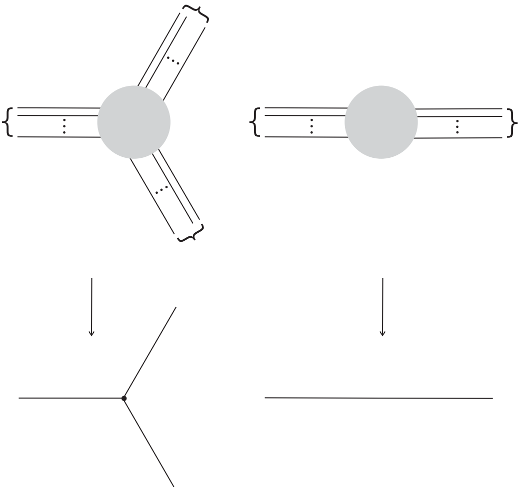

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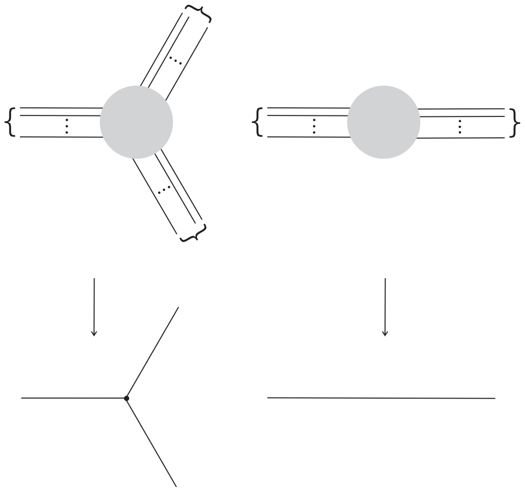
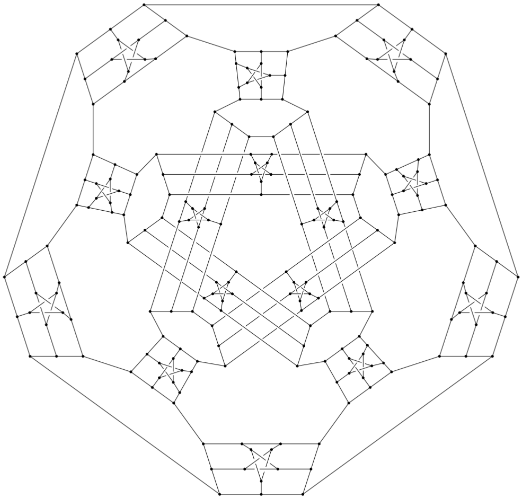


Descartes 1948; Adelson-Velskii & Titov 1973; Fiol 1991; Kochol 1996.

Example of superposition: Descartes snark (1948)



Edge-colourings as flows

A 3-edge-colouring of a cubic graph G can be thought of as a mapping

$$\phi\colon E(G) \to \mathbb{Z}_2 \times \mathbb{Z}_2 - 0 = \{01, 10, 11\}$$

such that the sum of colours around each vertex $= 0$.

$\Longrightarrow$

3-edge-colouring $=$ nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_2$ -flow

Superposition mapping

Let G and H be graphs.

A superposition mapping $f\colon G \to H$ is a mapping
from a subdivision G' of G to a subdivision H' of H s.t.

- vertex $\mapsto$ vertex
- edge $\mapsto$ edge or vertex (edge can be contracted to a vertex)
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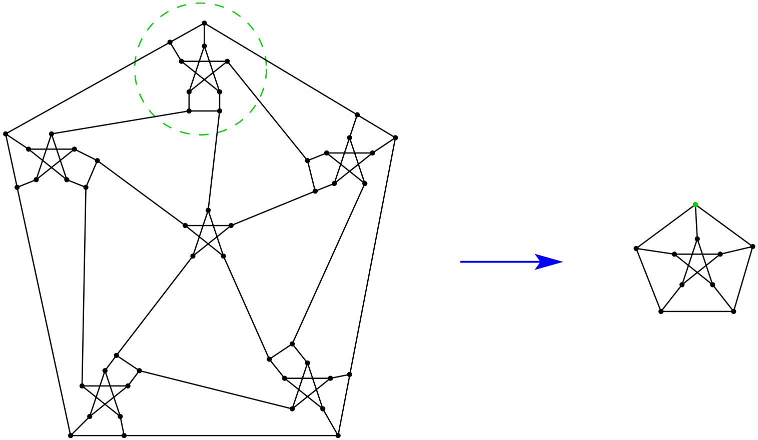
– $f\colon G' \to H'$ is onto

– G is cubic, but H need not be

Superposition mapping



Superposition mapping and flows

Let $f\colon G \to H$ be a superposition mapping. Then

- every $\mathbb{Z}_2 \times \mathbb{Z}_2$-valuation ϕ of G induces a $\mathbb{Z}_2 \times \mathbb{Z}_2$-valuation ϕ_* of H
- if ϕ is a flow on G, then ϕ_* is a flow on H

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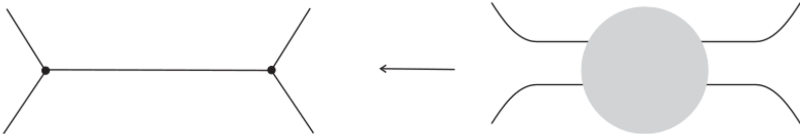
The aim is to contradict the existence of a nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_2$-flow ϕ on G provided H has no nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_2$-flow. For example:

- the induced valuation ϕ_* is nowhere-zero while H is a snark (contradiction!)
- the induced valuation ϕ_* fails to be a flow (contradiction!)

Dot product as superposition



$G.H$ = substitution of an edge of H with a dipole obtained from G

Superposition with active superedges

1. Choose a base snark H for superposition and a subgraph $X \subseteq H$, typically a circuit.
2. For each $e \in E(X)$ choose a snark K_e and create a superedge S_e by
   - removing two vertices
   - severing two edges, or by
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3. Replace each edge e on X with the superedge S_e
4. Add supervertices arbitrarily to obtain a cubic graph G

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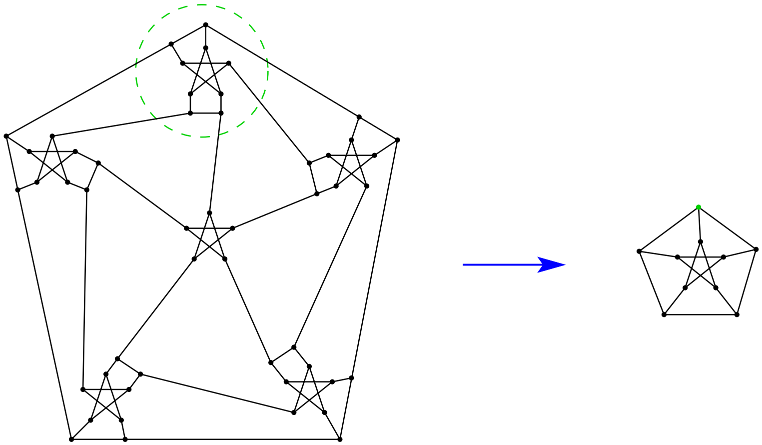
4. Add supervertices arbitrarily to obtain a cubic graph G
5. The choice of superedges guarantees that a nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_2$-flow on G would induce one on H (contradiction!).

Superposition with active supervertices



Applications of superposition (active superedges)

- Snarks with large girth [Kochol 1996]

There exists a cyclically 5*-connected snark of girth* g *for each* $g \geq 5$*.*

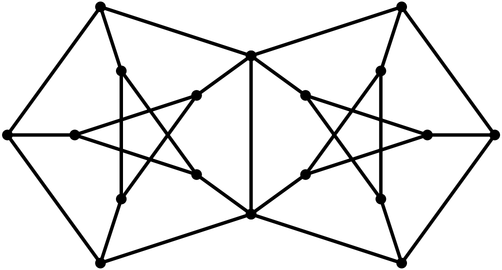
- Snarks with orientable polyhedral embeddings [Kochol 2009]

For every orientable surface S *of genus* ≥ 5 *there exists a cyclically* 5*-connected snark with a polyhedral embedding in* S*.*

- Snarks with given circular flow numbers [Máčajová & Raspaud 2006; Lukoťka & S. 2011]

For every rational number $r \in (4, 5]$ the exists a cyclically 4-edge-connected snark G with girth ≥ 5 for which $\Phi_c(G) = r$.

Base graph of superposition for $\Phi_c = 4 + 1/2$



A graph with $\Phi_c = 4 + 1/2$

Binary snarks

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A binary snark has a spanning tree T with all vertices of degree 3 or 1, and all leaves at the same distance from the centre r.

Equivalently:

T consists of three isomorphic binary trees whose roots are joined to the central vertex r

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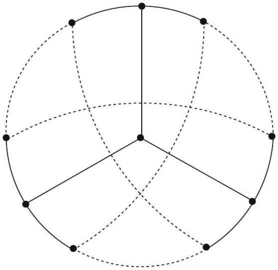

Binary snarks

Binary snarks appeared in connection with the study of homeomorphically irreducible spanning trees in cubic graphs (hists) [Hoffmann-Ostenhof, Ozeki, C.-Q. Zhang, . . .]

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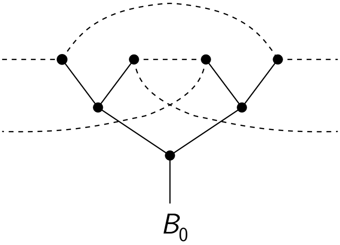
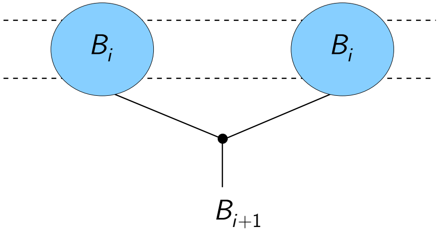
Conjecture (Hoffmann-Ostenhof & Jatschka 2017)

There exist infinitely many binary snarks with rotation property.

Binary snarks (active supervertices)





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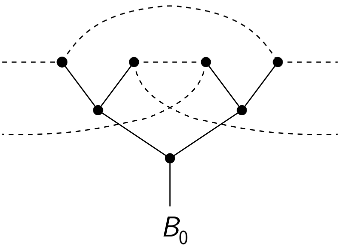
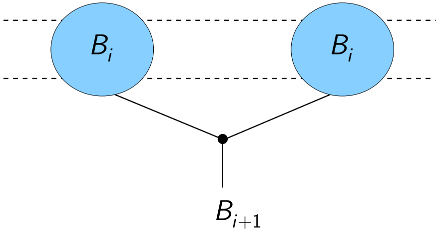
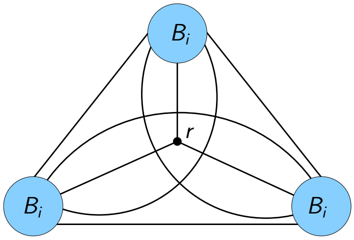


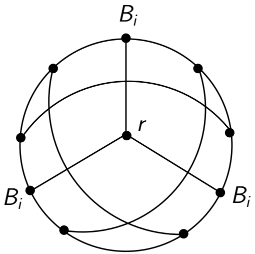


Every 3-edge-colouring of B_i, $i \geq 1$, assigns different colours to both pairs of dangling edges.

Binary snarks (active supervertices)



$\longrightarrow$



Even negator superposition

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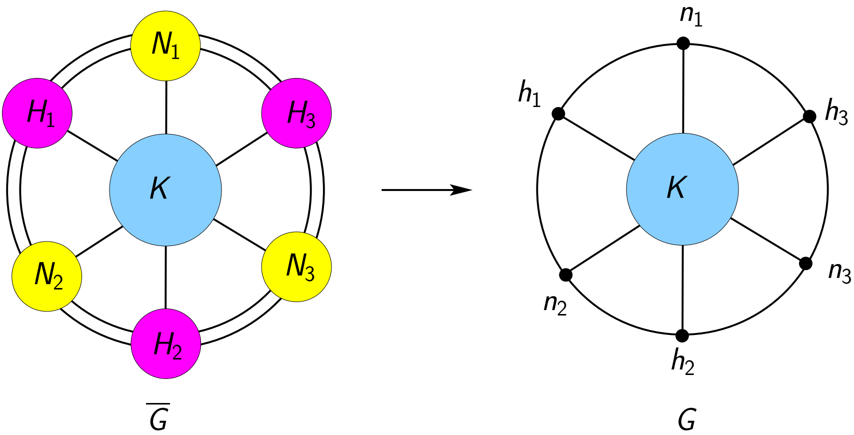
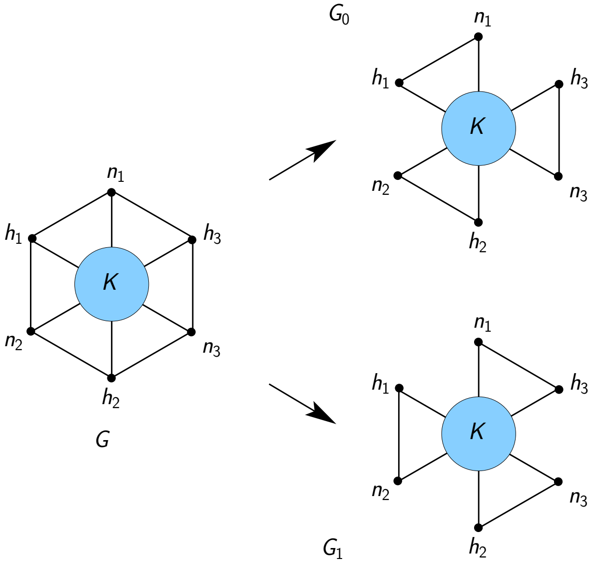


Even negator superposition



Even negator superposition

Theorem (Máčajová & S. 2019+)

Let G be a connected cubic graph and let $\overline{G}$ be created from G by an even negator superposition. If both G_0 and G_1 are snarks, then so is $\overline{G}$.

Furthermore, if every $(2,2,1)$-pole used in the superposition $\overline{G}$ is amiable, then $\overline{G}$ is a snark $\Longleftrightarrow$ both G_0 and G_1 are snarks.

A $(2,2,1)$-pole M is amiable if for every 2-connector S of M there exists a 3-edge-colouring of M s.t. both edges of S receive the same colour.

Application to permutation snarks

A permutation snark is a connected cubic graph of Class 2 with a 2-factor consisting of two chordless circuits.

Theorem (Máčajová & S. 2019+)

There exists a cyclically 5-edge-connected snark of order n for each $n \equiv 2 \pmod 8$ with $n \geq 34$.

Previously such snarks were known only for $n \equiv 10 \pmod{24}$ [Hägglund, Hoffmann-Ostenhof 2017].

Thank you for listening