Generating Graph and Matroid Minors

Sandra Kingan

Brooklyn College, CUNY

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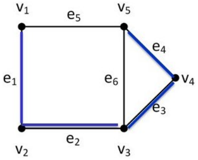
1. From graphs to matroids
2. From matroids to graphs
3. Growing graphs (and matroids) with a specific minor

1. From graphs to matroids

Consider the graph



It has incidence matrix

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\ \left[\begin{array}{cccccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{array}\right] \end{array}$$

Doing elementary row operations we get

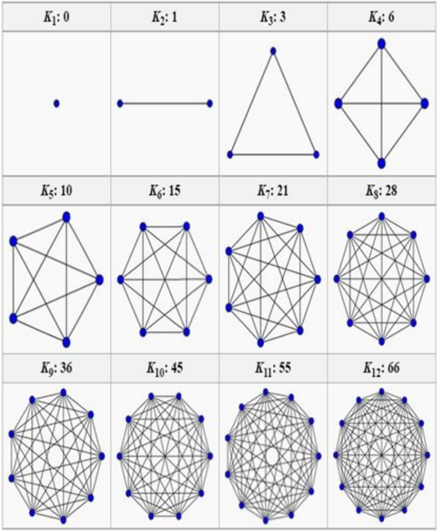
$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\ \left[\begin{array}{cccccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \end{array}\right] \end{array}$$

Delete the redundant last row of zeros to get the "matroid matrix."

A graph is a special kind of $\{0,1\}$-matrix.

A binary matroid is <u>any</u> $\{0,1\}$ matrix.

A rank r graph (with $n = r + 1$ vertices) is a substructure of K_n



The size of K_n is $\binom{n}{2}$.

A (simple) **binary matroid** is a substructure of the rank r binary projective geometry PG(r-1, 2).

PG(2,2)

1	2	3	4	5	6	7
1	0	0	0	1	1	1
0	1	0	1	0	1	1
0	0	1	1	1	0	1

PG(3,2)

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	0	0	0	0	0	0	0	1	1	1	1	1	1	1
0	1	0	0	0	1	1	1	0	0	0	1	1	1	1
0	0	1	0	1	0	1	1	0	1	1	0	0	1	1
0	0	0	1	1	1	0	1	1	0	1	0	1	0	1

PG(4,2)

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	1	0	0	0	0	0	0	0	1	1	1	1	1	1	1
0	0	1	0	0	0	1	1	1	0	0	0	1	1	1	1
0	0	0	1	0	1	0	1	1	0	1	1	0	0	1	1
0	0	0	0	1	1	1	0	1	1	0	1	0	1	0	1

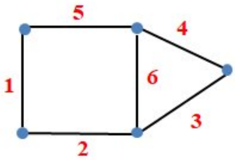
17	18	19	20	21	22	23	24	25	26	27	28	29	30	31
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
0	0	0	0	0	0	0	1	1	1	1	1	1	1	1
0	0	0	1	1	1	1	0	0	0	0	1	1	1	1
0	1	1	0	0	1	1	0	0	1	1	0	0	1	1
1	0	1	0	1	0	1	0	1	0	1	0	1	0	1

PG(5,2)

7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1
0	0	0	0	1	1	1	1	1	1	1	0	0	0	0	0	0	0	1	1	1	1	1	1
0	1	1	1	0	0	0	1	1	1	1	0	0	0	1	1	1	1	0	0	0	0	1	1
1	0	1	1	0	1	1	0	0	1	1	0	1	1	0	0	1	1	0	0	1	1	0	0
1	1	0	1	1	0	1	0	1	0	1	1	0	1	0	1	0	1	0	1	0	1	0	1

31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50	51	52	53
0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1
1	1	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	0	0	0	0	0	0
1	1	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1
1	1	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0
0	1	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1

The size of $PG(r-1,2)$ is $2^r - 1$.

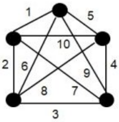


$$A = \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}_{F_2} \end{array}$$

I am asked sometimes what a matroid is. I often revert to our sacred writings and recall the encounter of Alice with the grinning Cheshire cat. At one stage the cat vanishes away, beginning with the tip of its tail and ending with the grin, which persists long after the remainder of the cat. - William Tutte

Definition 1.

A **minor** of a graph is obtained by deleting edges (and any isolated vertices) and contracting edges.



$$
\begin{array}{cccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
\end{array}
$$
$$
\begin{bmatrix}
1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1
\end{bmatrix}
$$

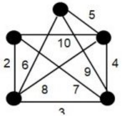
K_5

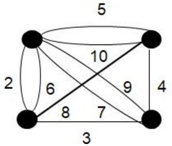
$$
\begin{array}{cccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
\end{array}
$$
$$
\begin{bmatrix}
1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1
\end{bmatrix}
$$

$K_5\backslash 1$

$$
\begin{array}{cccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
\end{array}
$$
$$
\begin{bmatrix}
1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1
\end{bmatrix}
$$

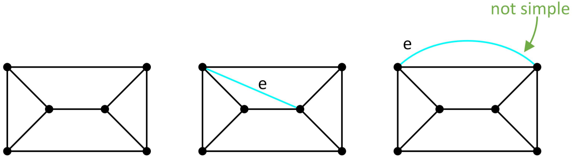
$K_5/1$



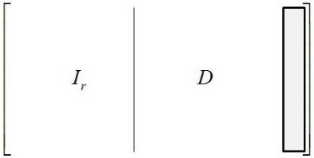


Definition 2a.

G with an edge e added between non-adjacent vertices is denoted by $G + e$ and called a **(simple) edge addition** of G.



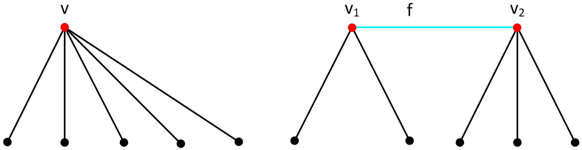
G G+e



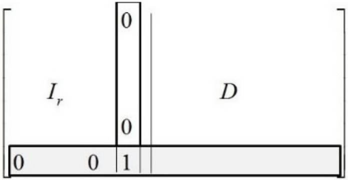
Simple extension

Definition 2b.

Suppose G is a graph with a vertex v such that $deg(v) \geq 4$. To **split** vertex v, replace v with two distinct adjacent vertices v_1 and v_2 and join each neighbor of v to exactly one of v_1 or v_2 so that $deg(v_1) \geq 3$ and $deg(v_2) \geq 3$.



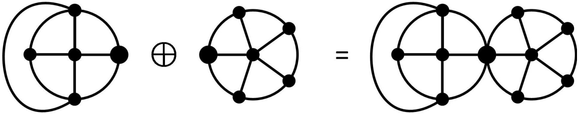
When a vertex is split, two new vertices and a new edge f between them are introduced. The resulting graph is denoted by $G \circ f$.

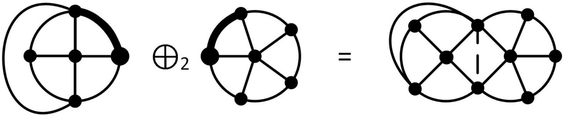


Definition 3.

A graph is **3-connected** if at least 3 vertices must be removed to disconnect the graph.

A graph that is not 3-connected can be constructed from its 3-connected proper minors using 1-sums and 2-sums.



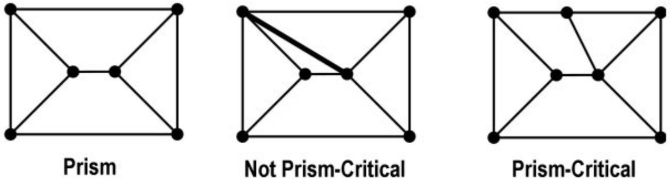


Suppose G and H are simple 3-connected graphs and G has an H-minor.

Definition 4.

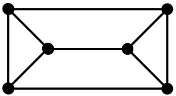
(a) An edge e in G is H-**deletable** if $G \backslash e$ is 3-connected and has an H-minor.

(b) G is H-**critical** if G has no H-deletable edges.

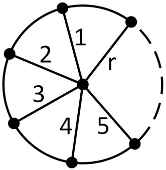


Prism-Free Graphs (Dirac 1963)

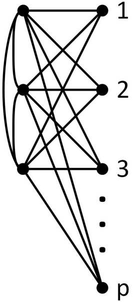
A simple 3-connected graph G has no prism minor if and only if G is isomorphic to $K_5 \backslash e$, K_5, W_{n-1}, for $n \geq 4$, $K_{3,n-3}$, $K'_{3,n-3}$, $K''_{3,n-3}$, or $K'''_{3,n-3}$, for $n \geq 6$.



Prism



W_r



2. From matroids to graphs

Wheels Theorem (Tutte 1961)

Suppose G is a simple graph that is not a wheel. Then G is 3-connected if and only if there is a sequence of simple 3-connected graphs $G_0, \ldots, G_t$ such that G_0 is a wheel, $G_t \cong G$, and for $1 \leq i \leq t$ either $G_i = G_{i-1} + e$ or $G_i = G_{i-1} \circ f$.

Splitter Theorem (Seymour 1980 Negami 1982)

Suppose G and H are simple 3-connected graphs such that G is not a wheel and $H \ncong W_3$. Then G has an H-minor if and only if there exists a sequence of simple 3-connected graphs $G_0, \ldots, G_t$ such that $G_0 = H$, $G_t \cong G$, and for $1 \leq i \leq t$ either $G_i = G_{i-1} + e$ or $G_i = G_{i-1} \circ f$.

Main Theorem (Costalonga and SRK 2019+)

Suppose G and H are simple 3-connected graphs, where $|E(G)| \geq |E(H)| + 3$, and $|V(G)| \geq |V(H)| + 1$. If G has an H-minor, then there exists a set of H-deletable edges D such that

$$|D| \geq |E(G)| - |E(H)| - 3[|V(G|) - |V(H)|]$$

and a sequence of H-critical graphs

$$G_{|V(H)|}, \ldots, G_{|V(G)|},$$

where $G_{|V(H)|} \cong H$, $G_{|V(G)|} = G \backslash D$, and for all i such that $|V(H)| + 1 \leq i \leq |V(G)|$:

(i) $G = G' \circ f$;

(ii) $G = G' + e \circ f$, where e and f are incident to a degree 3 vertex; or

(iii) $G = G' + \{e_1, e_2\} \circ f$, where e_1, e_2, f are incident to a degree 3 vertex.

This is the main result in a paper titled "Minor-preserving deletable edges" on arxiv.

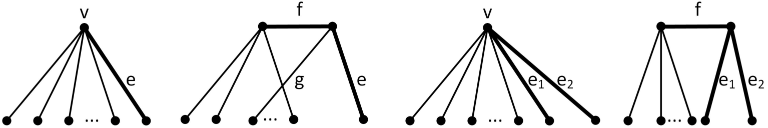
Key Lemma:

Suppose G and H are simple 3-connected graphs, where G is not a wheel and $H \ncong W_3$. If G is H-critical, then there is an H-critical graph G' on $|V(G)| - 1$ vertices such that:

(i) $G = G' \circ f$;

(ii) $G = G' + e \circ f$, where e and f are incident to a degree 3 vertex; or

(iii) $G = G' + \{e_1, e_2\} \circ f$, where e_1, e_2, f are incident to a degree 3 vertex.

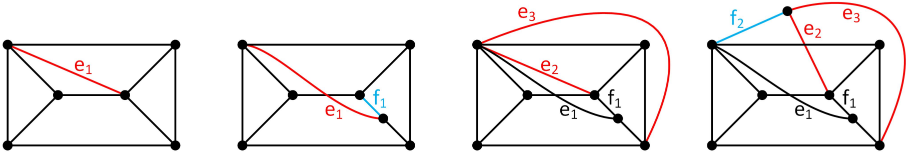


Vertex split of G+e

Vertex split of G+$\{e_1, e_2\}$

The third operation is necessary.

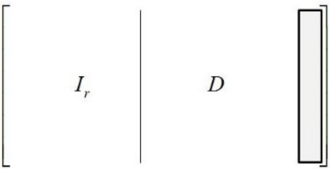
Consider the graph $G = prism + e_1 \circ f_1 + \{e_2, e_3\} \circ f_2$ constructed as shown below. It is minimally 3-connected.



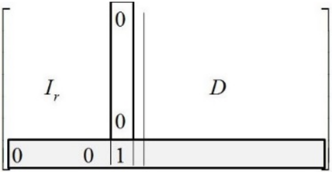
Prism+e_1 Prism+$e_1 \circ f_1$ Prism+$e_1 \circ f_1$+$\{e_2,e_3\}$ Prism+$e_1 \circ f_1$+$\{e_2,e_3\} \circ f_2$

An exhaustive search and construction of all graphs with 8 vertices and 14 edges shows that the only way to obtain G is following the order of operations shown above.

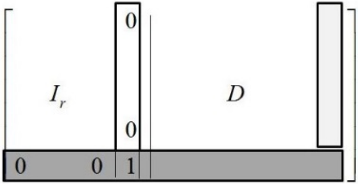
Easier to see in binary matrices than graphs, but you have to look at it just right



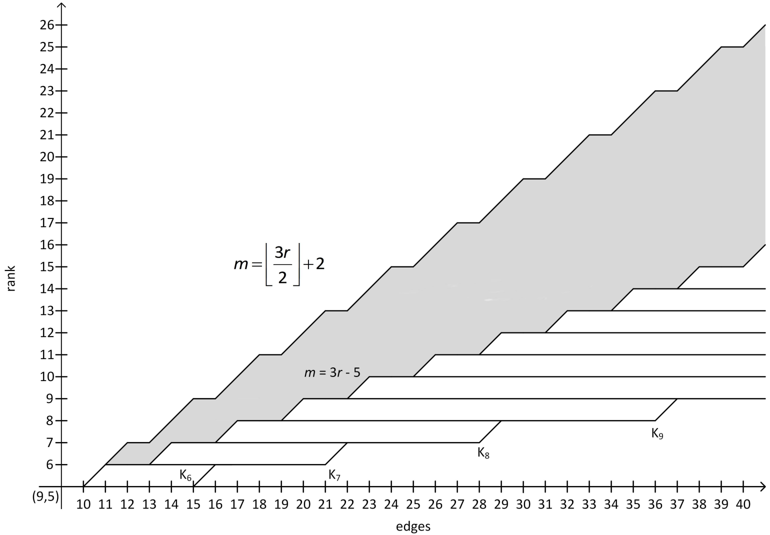
Simple extension



Cosimple coextension



Cosimple coextension of a extension



rank
edges
(9,5)
$m = \left\lfloor \frac{3r}{2} \right\rfloor + 2$
m = 3r - 5
K_6
K_7
K_8
K_9

Start with a 3-connected graph J. The three operations of Theorem 1 are:

(i) $J \circ f$;

(ii) $J + e \circ f$, where e and f are incident to a degree 3 vertex; or

(iii) $J + \{e_1, e_2\} \circ f$, where e_1, e_2, f are incident to a degree 3 vertex and e_1, e_2 are not in a triangle.

(1) How many vertex splits of J are there of the first type?

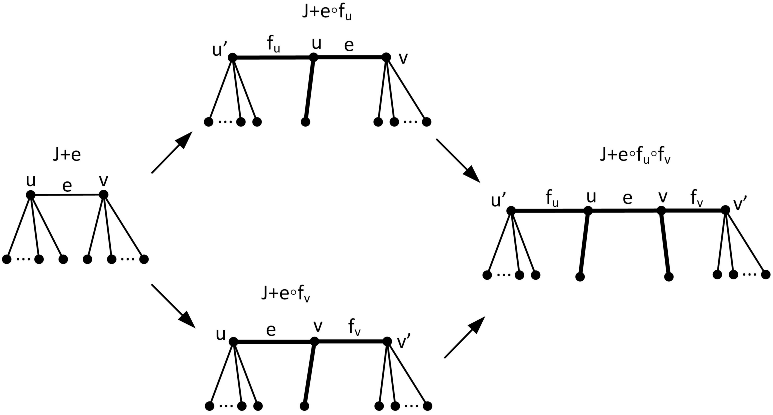
Since each vertex of degree at least 4 may be split, the maximum number of splitting operations is $\sum(deg(v_i) - 3) = 2|E(J)| - 3|V(J)|$

(3) How many vertex splits per graph $J + e$ of the third type?

One

(2) How many vertex splits of the second type?

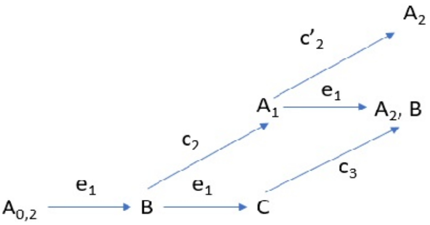
Two (Why?)



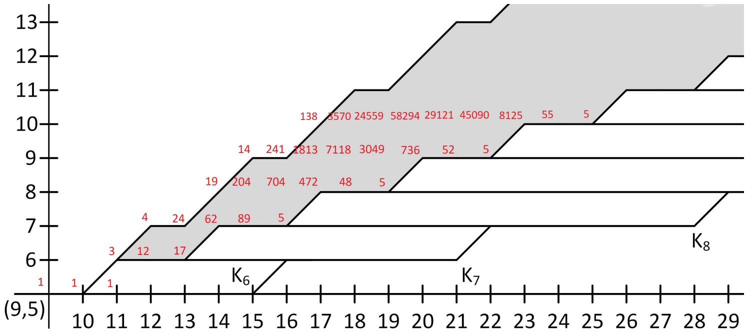
Vertex-Splits Theorem

Suppose J is a simple 3-connected graph and let u and v be non-adjacent vertices in J. Then the only J-critical vertex splits of $J + e$ are $J + e \circ f_u$, $J + e \circ f_v$, and $J + e \circ \{f_u, f_v\}$, where $e = uv$ and $\{e, f_u\}$ and $\{e, f_v\}$ are in distinct triads. Moreover, no other vertex split of $J + e$ is J-critical.

3. Algorithm



C_1: Split all possible vertices.
E_1: Add an edge. Generate one graph for every possible non-adjacent pair.
C_2: Split a vertex adjacent to the edge added by E_1 in every possible way.
C_2': Split the vertex at the other end of the edge added by E_1 (that is not the one split by C_2) in every way possible.
C_3: If the edges added by two applications of E_1 are adjacent, split the common vertex to form a triad.



$$\{Prism\ critical\ graphs\} = \{Minimally\ 3\ connected\ graphs\} - \{W_r, K_{3,r-2}\}$$

Theorems and Algorithm hold for arbitrary H.

Rank	Number of 3-connected simple graphs (A006290)	Generators	Number of minimally 3-connected graphs from (A199676)
3	1		1
4	3		1
5	17	2	3
6	136	32	5
7	2388	184	18
8	80890	1452	57
9	5114079	13028	285
10	573273505	168957	1513
11	113095167034		9824
12	39582550575765		
13	24908445793058442		
14	28560405143495819079		
15	60364410130177223014724		
16	237403933018799958309530349		
17	1750323137355778190158082029500		
18	24333358813699371350715221107464003		
19	640811613278752754485012443963579501421		

Tutte (1963). A census of planar maps. Can. J. Math, 15, 249 - 271.

R. W. Robinson and T. R. S. Walsh (1993). Inversion of cycle index sum relations for 2- and 3-connected graphs, J. Combin. Theory Ser. B. 57, 289-308.

D. Wind (2011) Connected graphs with fewest spanning trees

THANK YOU FOR YOUR ATTENTION!