Graphs with specified degrees

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A problem: Count regular graphs

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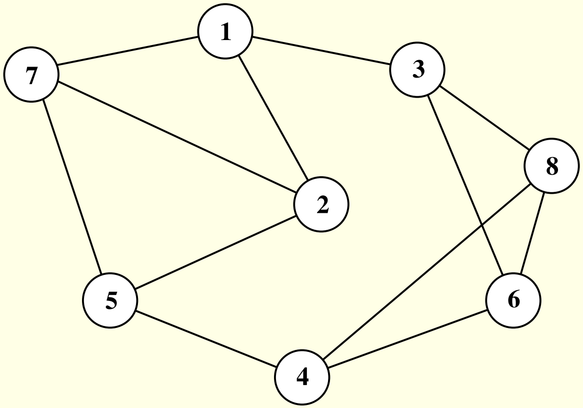

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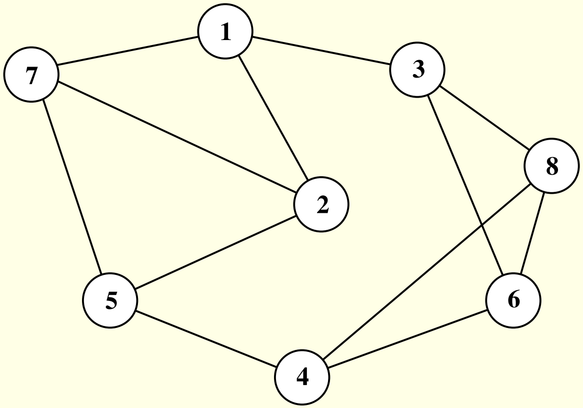

A graph that contributes to $RG(8, 3) = 19355$

The numbers grow rather quickly, for example

$$\mathrm{RG}(30, 9) = 1833667837354213051325773436838378261912912280930223000342112435635482956708281040928263924915.$$

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Our interest is in asymptotic counting, where we want a good approximation for $\mathrm{RG}(n, d)$ as $n \to \infty$ while $d = d(n)$.

We can assume $1 \leq d \leq (n-1)/2$, since $d = 0$ is trivial and $d > (n-1)/2$ follows by complementation. Also, nd is even.

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The case of $\mathrm{RG}(n, 3)$ was solved by Ron Read in his 1958 thesis at the University of London.

Nothing much then happened for 20 years, until Bender and Canfield, and independently Wormald, solved it for arbitrary constant d.

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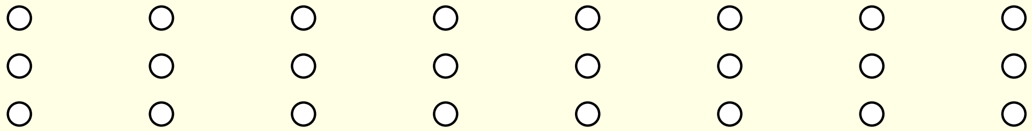

Take 8 groups of 3 dots each.

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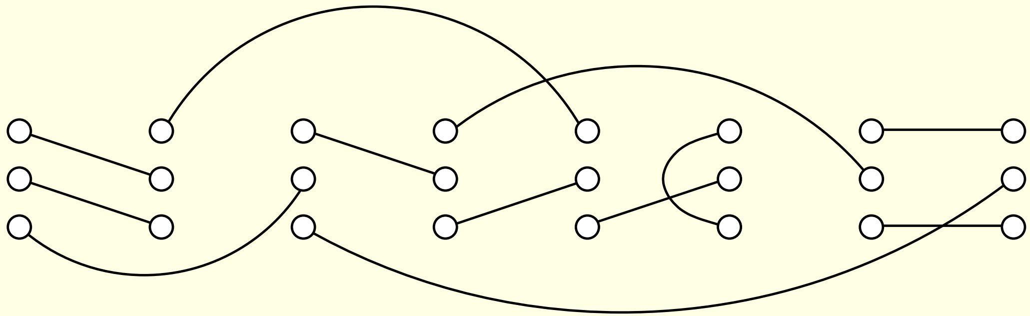

Pair the 24 dots together somehow.

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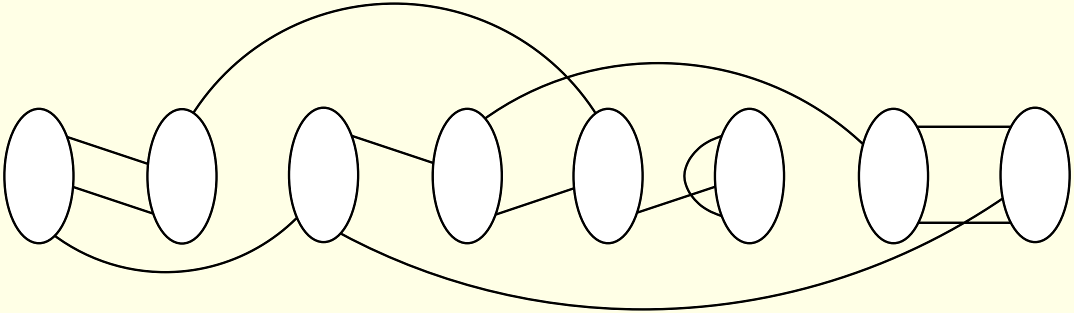

Convert the groups of dots into vertices.

Note the loops and multiple edges.

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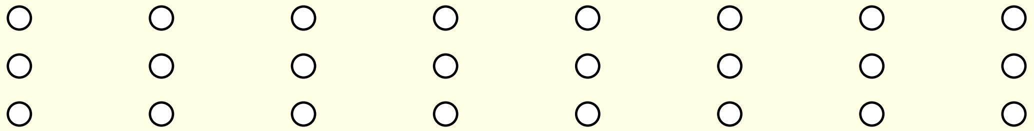

Try again: Take groups of dots.

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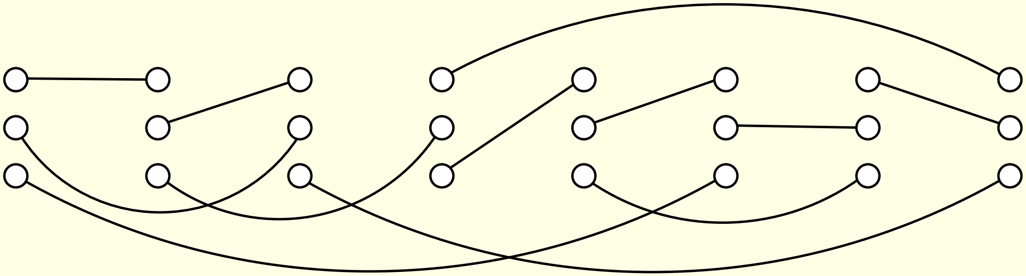

Pair them somehow.

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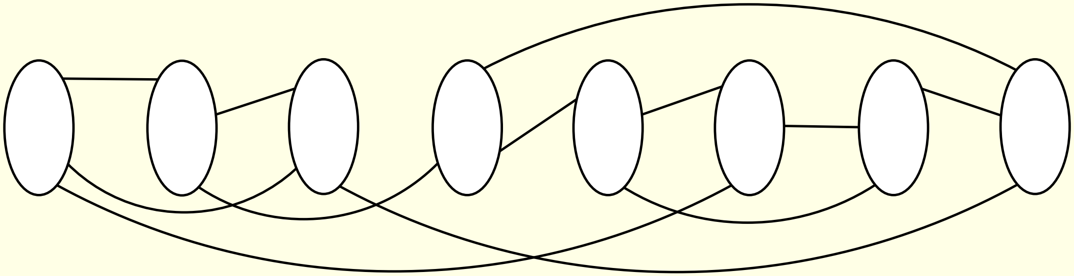

This time the result is a simple regular graph.

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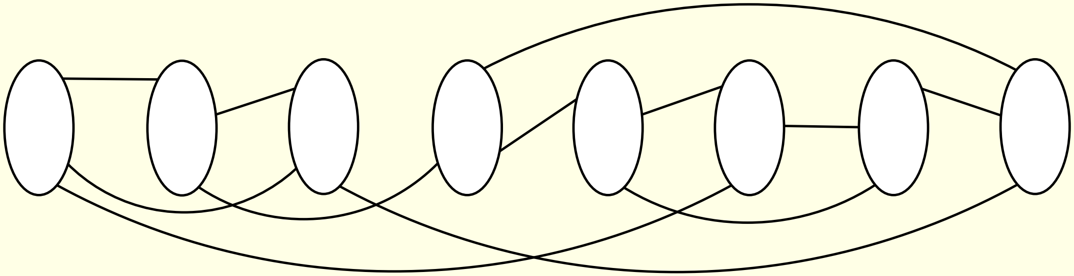

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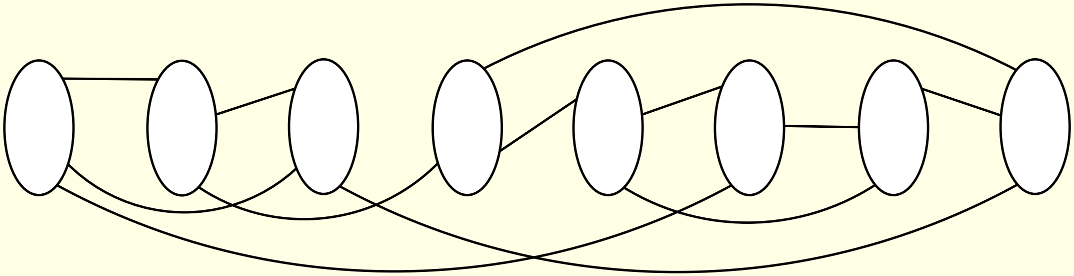

The pairing model is the most important tool for counting and studying regular graphs of low degree.

The pairing model used for counting

Consider pairings for order n and degree d.

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Conclusion

$$\mathrm{RG}(n,d) = \frac{(nd)!}{(nd/2)!\,2^{nd/2}\,(d!)^n}\,P(n,d),$$

where $P(n,d)$ is the probability that a random pairing gives a simple graph.

The pairing model used for counting (continued)

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Result: $$P(n,d) \approx \exp\left(-\frac{d^2-1}{4}\right).$$

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$$\text{Result:} \qquad P(n, d) \approx \exp\Bigl(-\frac{d^2 - 1}{4}\Bigr).$$

If we attempt to let $d \to \infty$ too quickly, the terms in the inclusion-exclusion series become extremely large compared to the sum of the terms, so it becomes increasingly difficult to get a good estimate.

The pairing model and switchings

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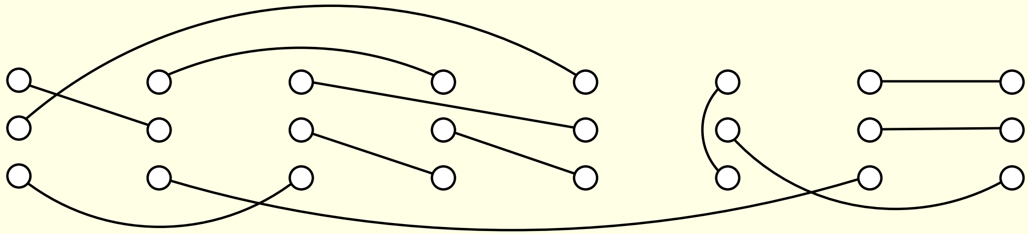
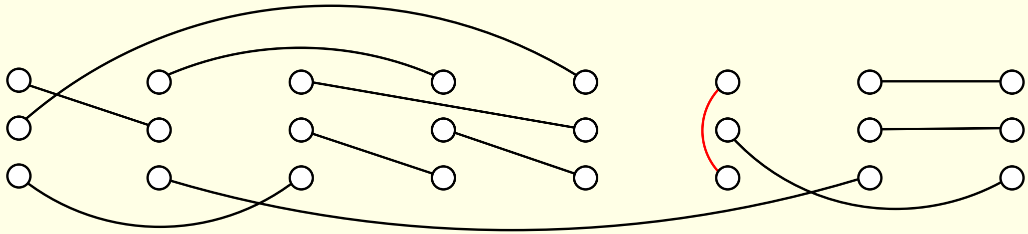


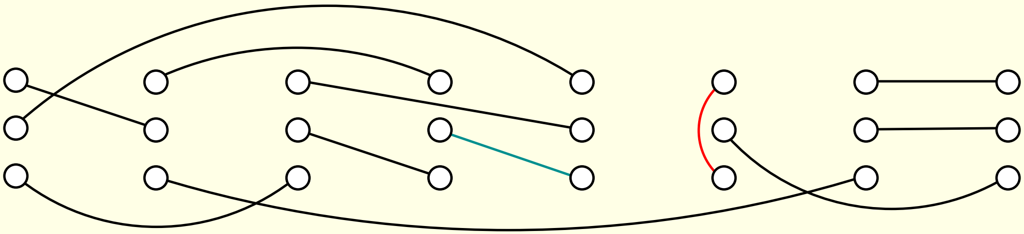
Consider a pairing.

The pairing model and switchings



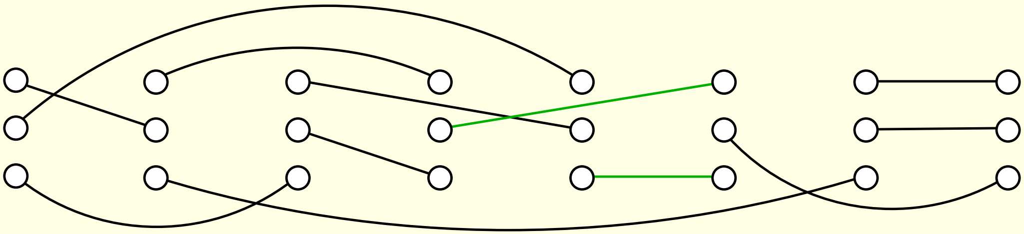
There is a loop.

The pairing model and switchings



Choose some other edge.

The pairing model and switchings



Switch those two edges with another two.

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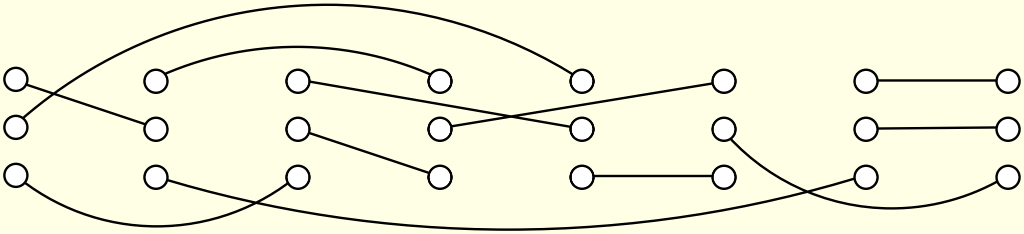
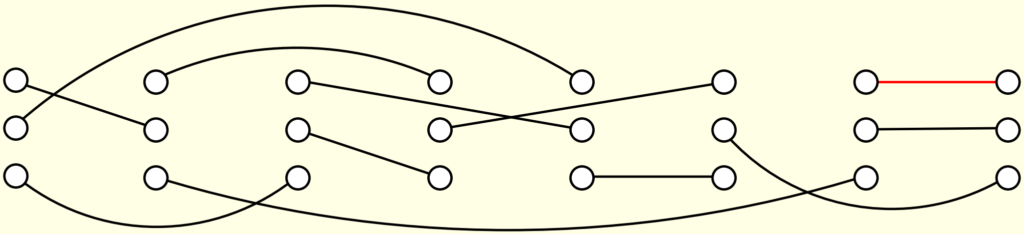


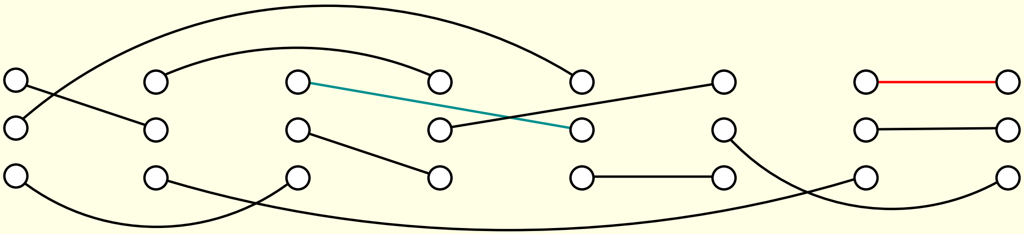
Now the loop is gone.

The pairing model and switchings



But there is still a double edge.

The pairing model and switchings



Choose some other edge.

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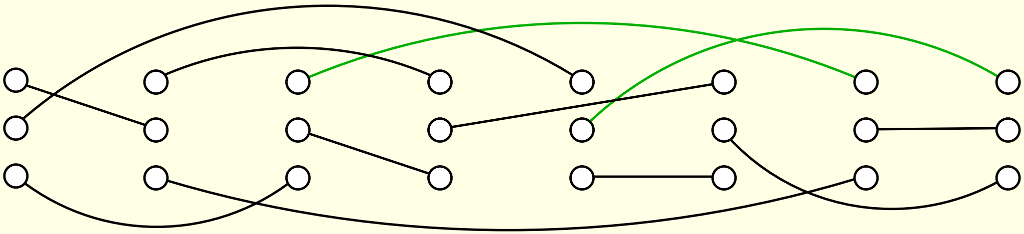

Switch two edges.

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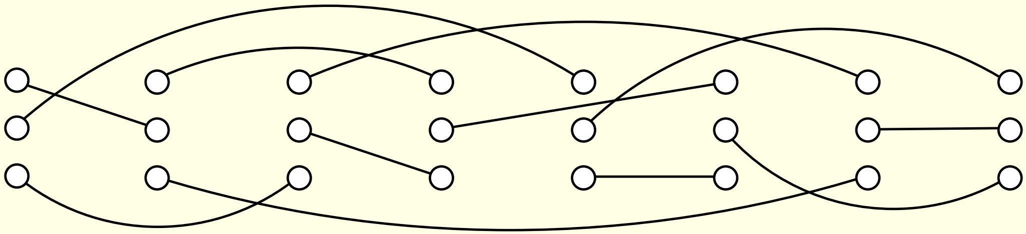

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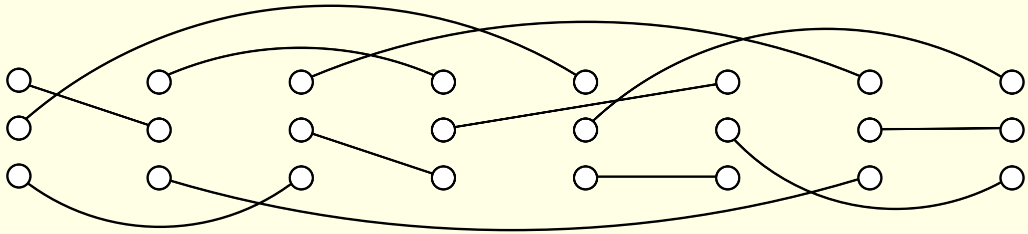

Now we have a simple graph.

Let $N(s,t)$ be the number of pairings with s double edges and t loops.
Using switchings we get estimates of

$$\frac{N(s,t+1)}{N(s,t)} \quad \text{and} \quad \frac{N(s+1,0)}{N(s,0)}$$

for significant s,t.
From this we can derive a positive term series for $1/P(n,d)$.

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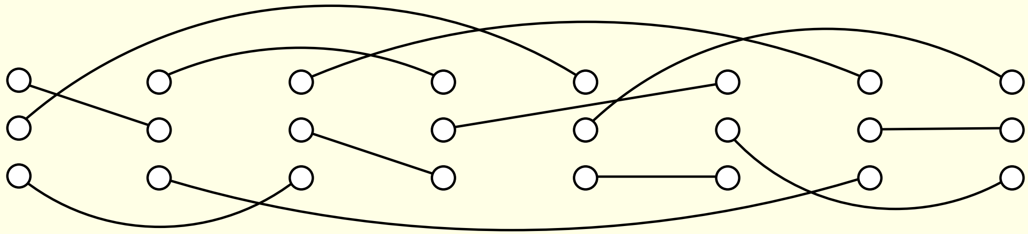

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Result: Same formula, for $d = o(n^{1/3})$. (McKay, 1985)

The pairing model and switchings (continued)

In 1991, McKay and Wormald used switchings of 3 edges to prove that

$$\mathrm{RG}(n,d) = \frac{(nd)!}{(nd/2)!\,2^{nd/2}\,(d!)^n}\,\exp\biggl(-\frac{d^2-1}{4} - \frac{d^3}{12n} + o(1)\biggr)$$

for $d = o(n^{1/2})$.

Gao and Wormald (2016) improved the coverage of highly-irregular degree sequences.

But, how to make one graph uniformly at random?

Example: Simple graphs with degrees 4,3,3,4,3,3,3,3

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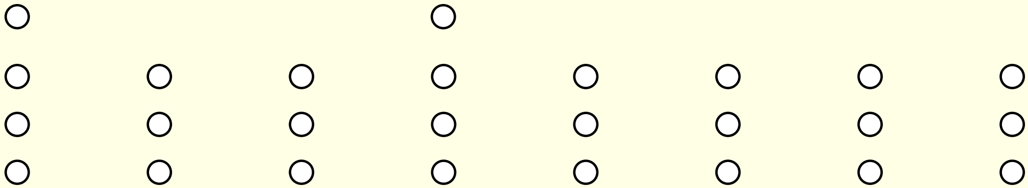

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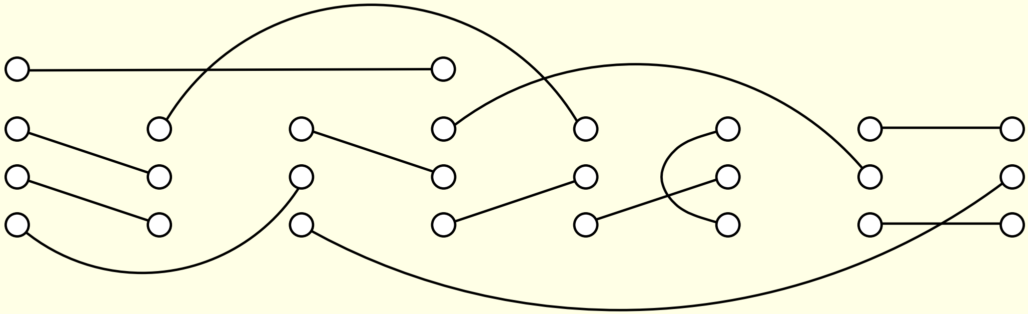

Pair them at random.

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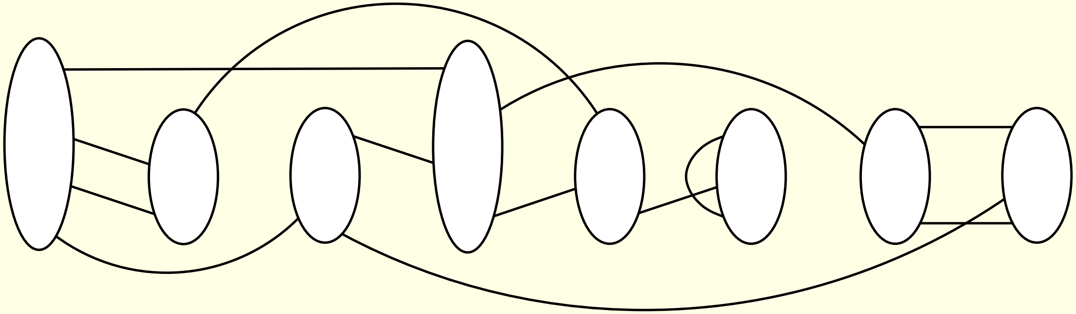

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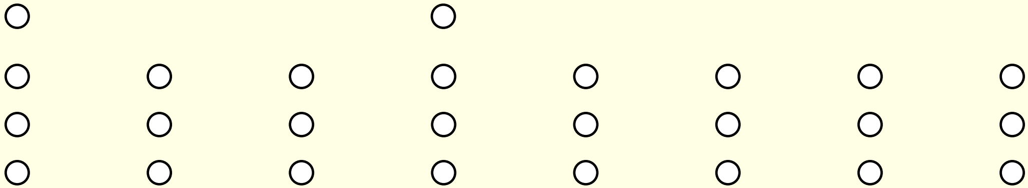

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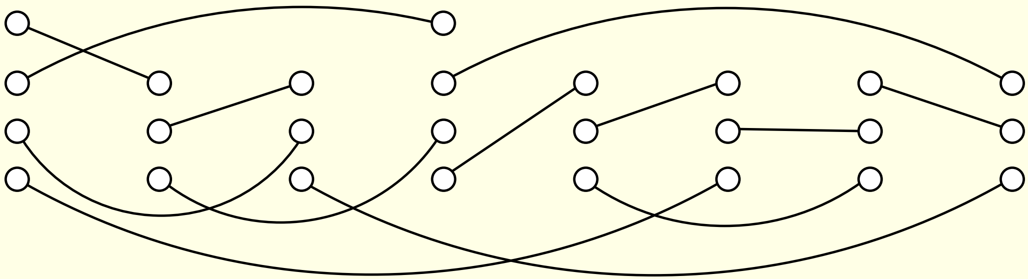

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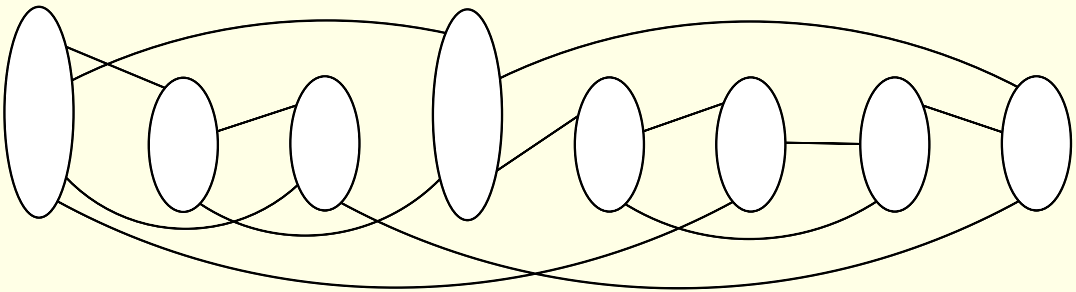

This time the result is simple.

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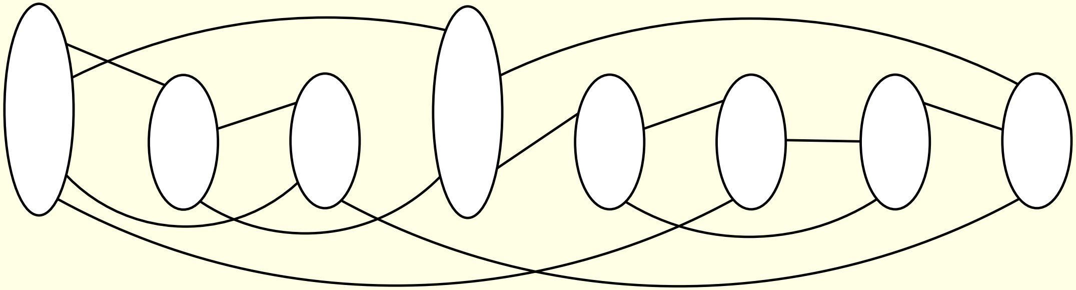

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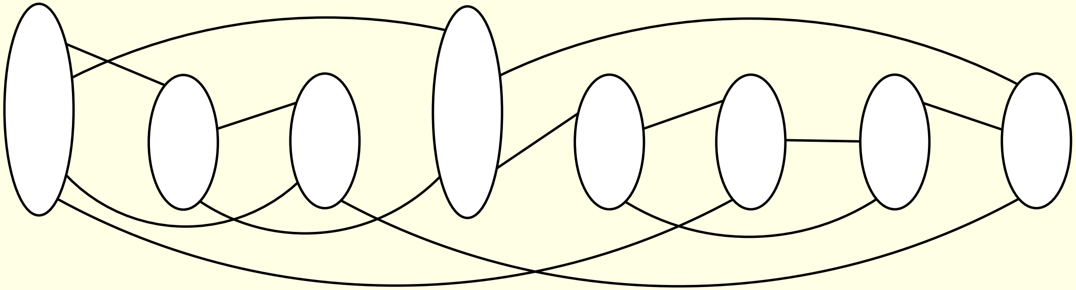

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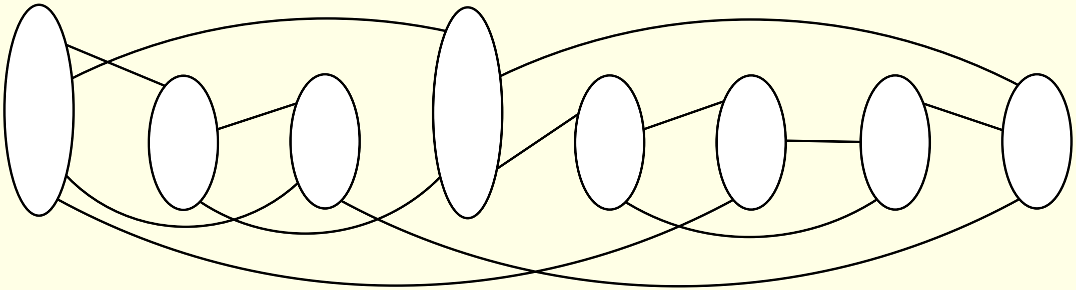

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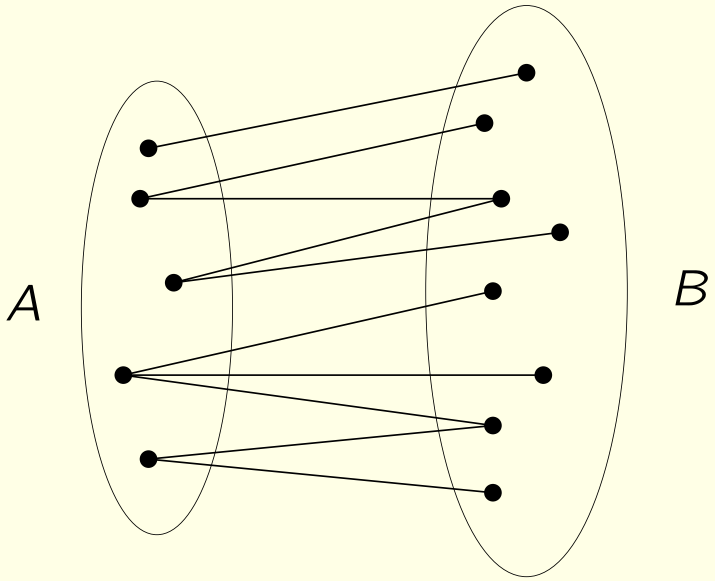
Alas, this is only efficient for low degree. For higher degree, too many attempts are required before a simple graph is obtained.

Accept-reject strategy

Consider two sets and a relation between them.



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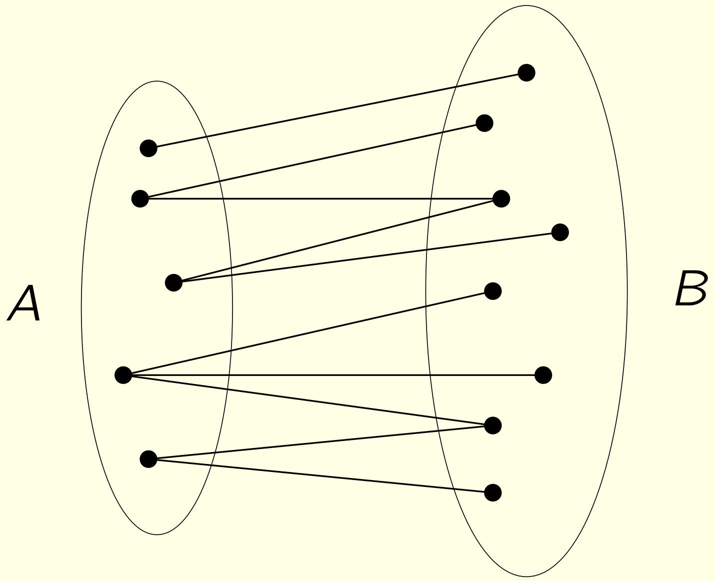

Suppose we know how to generate a random element of A.

How do we generate a random element of B?

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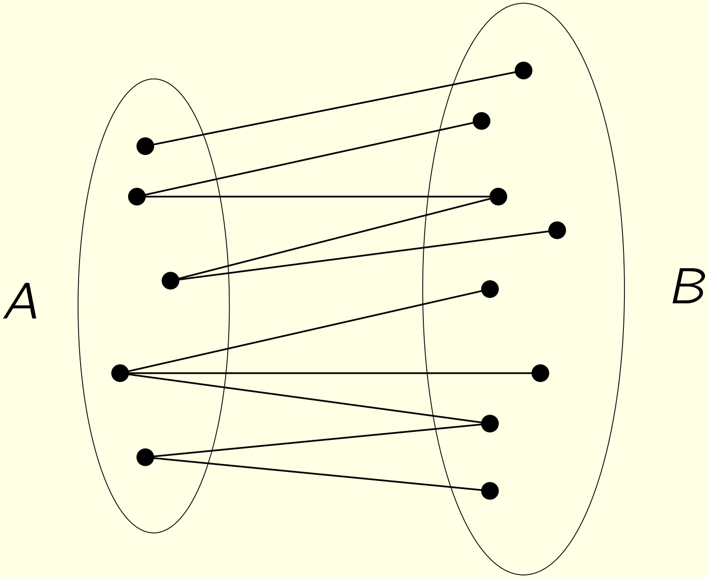

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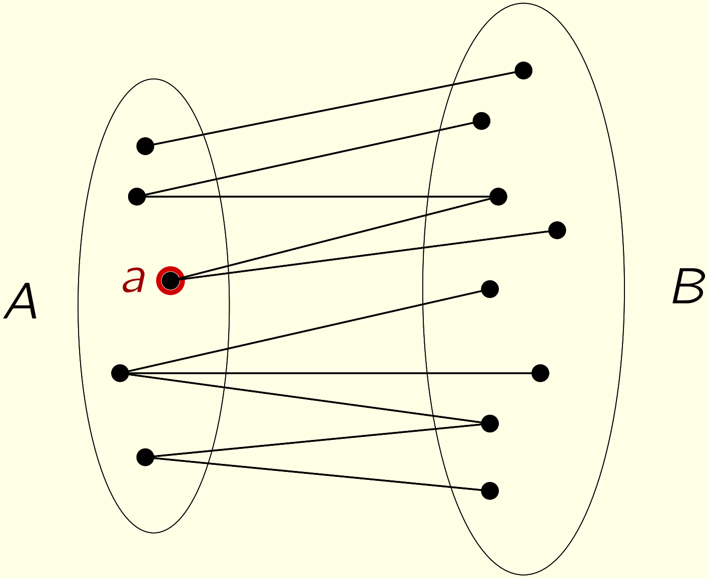
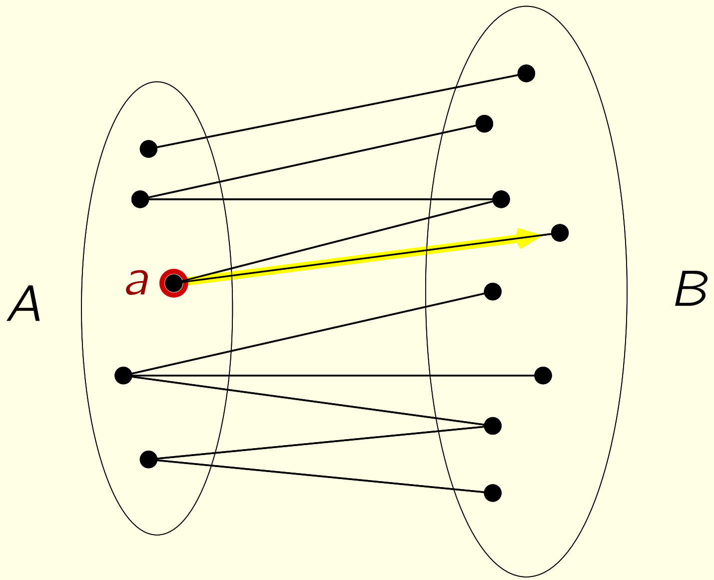


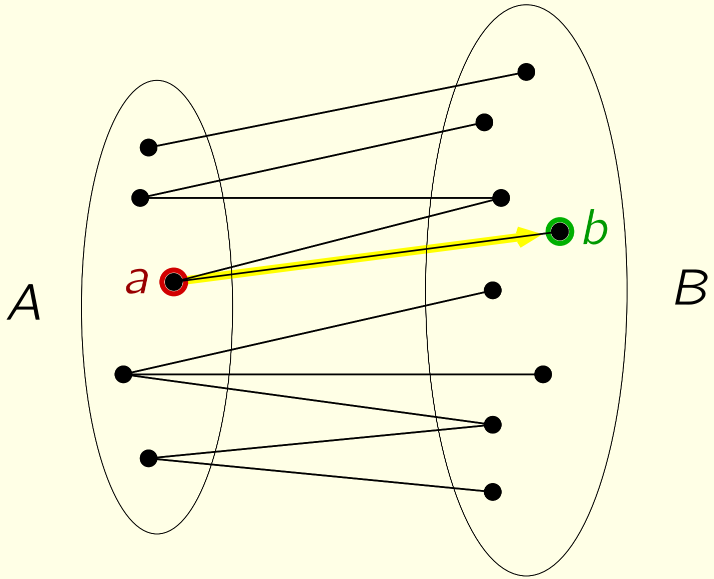
1. Choose random $a \in A$.

Accept-reject strategy



1. Choose random $a \in A$.
2. Take a random edge to B.

Accept-reject strategy



1. Choose random $a \in A$.
2. Take a random edge to B.
3. Accept $b \in B$ with probability proportional to $\deg(a)/\deg(b)$.
   If unsuccessful, try again.

Back to simple graphs with degrees 4,3,3,4,3,3,3,3

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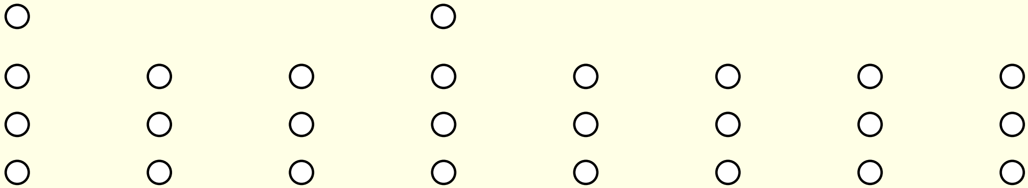
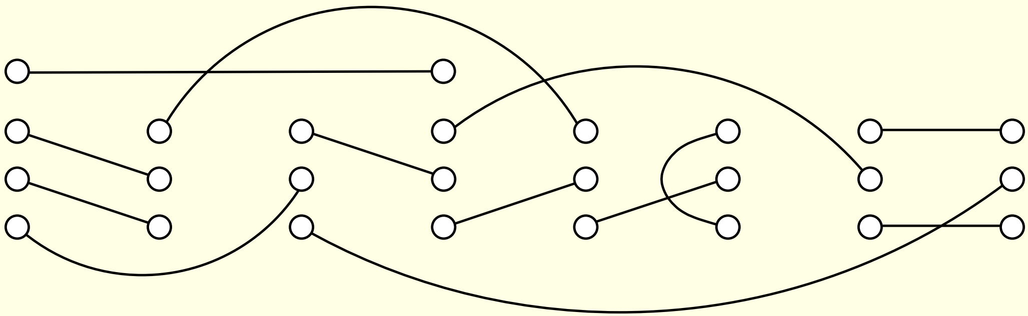


Take groups of dots according to the required degrees.

Back to simple graphs with degrees 4,3,3,4,3,3,3,3



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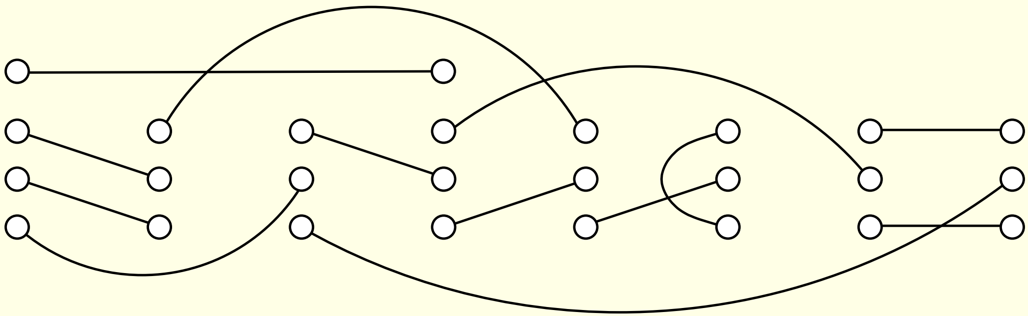

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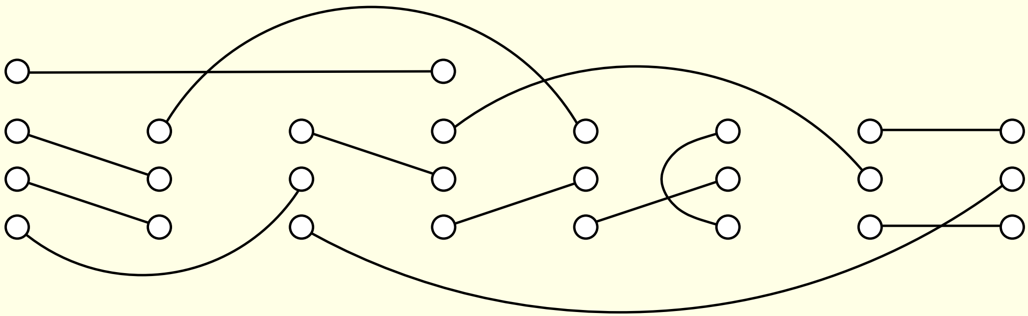

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Using an accept-reject strategy, we can transfer uniform randomness:

$$G(1,2) \to G(1,1) \to G(1,0) \to G(0,0)$$

and then we will have a uniformly random simple graph.

Simple graphs with degrees 4,3,3,4,3,3,3,3 (continued)

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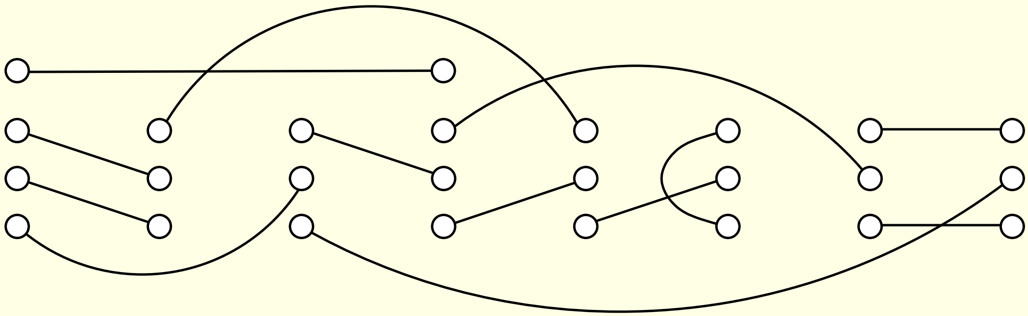
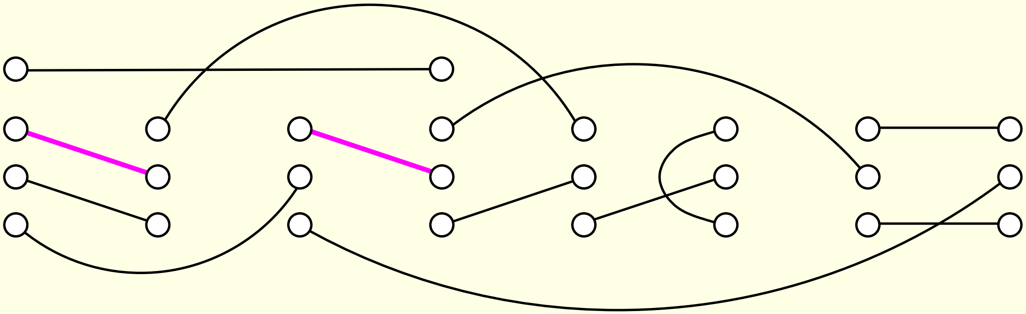


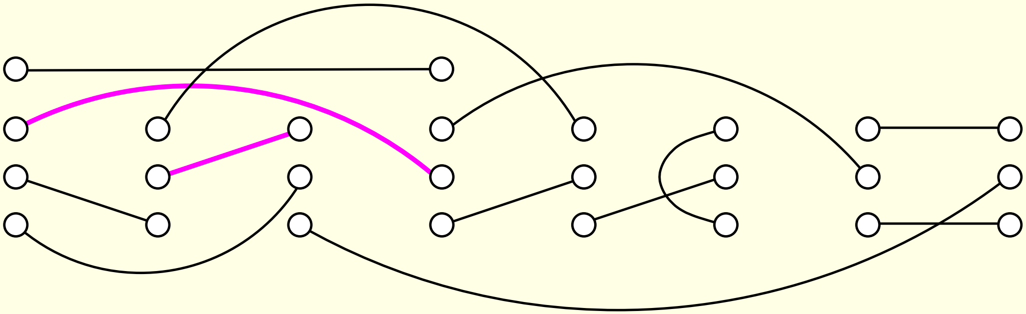
A random member of $G(2, 1)$.

Simple graphs with degrees 4,3,3,4,3,3,3,3 (continued)



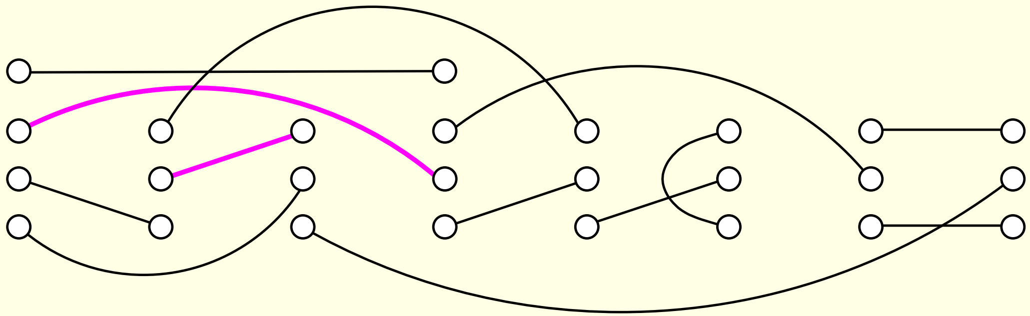
Choose an edge in a double edge and one other.

Simple graphs with degrees 4,3,3,4,3,3,3,3 (continued)



Swap for two other edges.

Simple graphs with degrees 4,3,3,4,3,3,3,3 (continued)



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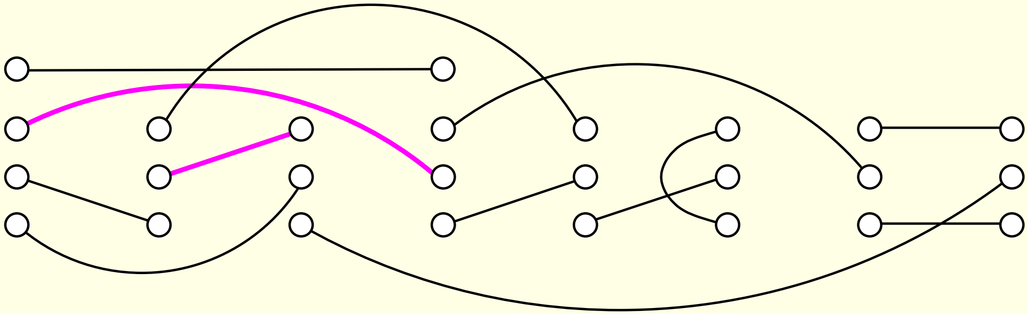

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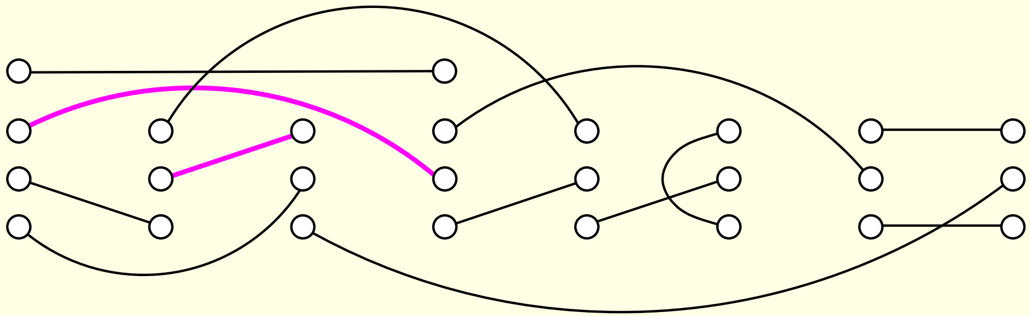

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Gao and Wormald substantially improved it and got to $d = o(n^{1/2})$.

There are no known polynomial expected-time algorithms to generate uniformly random regular graphs for degrees over $n^{1/2}$.

Iterative methods exist (e.g. Markov chains) that approach a uniform distribution asymptotically.

Random graphs with approximately the given degrees

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The Chung-Lu Model defines

$$p_{jk} = \frac{w_j w_k}{\sum_i w_i},$$

where $w_1, \ldots, w_n$ are some positive weights.

This is very simple to implement and easy to analyse.

It is not true that the probability of a graph depends only on its degree sequence.

The β-model of random graph

Let $\beta_1, \ldots, \beta_n$ be some real numbers and define

$$p_{jk} = \frac{e^{\beta_j + \beta_j}}{1 + e^{\beta_j + \beta_j}}.$$

This independent-edge model uniquely has the property that the probability of any graph depends only on its degree sequence.

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Now suppose we have a degree sequence $d_1, \ldots, d_n$ and further wish that the expectation of the degree of each vertex j is d_j. This gives

$$\sum_{k \neq j} p_{jk} = d_j, \qquad (1 \leq j \leq n). \qquad (*)$$

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Call this the **β-model for $d_1, \ldots, d_n$.**

The β-model for $d_1, \ldots, d_n$

Barvinok and Hartigan defined the δ-tame class of degree sequences. Approximately: $|\beta_j| \leq C$ for all j, for some constant C.

All degrees are $\Theta(n)$ but the variation can be great. For example all degree sequences with

$$0.25n \leq d_j \leq 0.74n \qquad (1 \leq j \leq n)$$

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Fix $Y \subseteq \binom{[n]}{2}$. Define two random variables:

$X = |E(G) \cap Y|$ when G is a uniformly random graph with degrees $d_1, \ldots, d_n$;

$X_\beta = |E(G) \cap Y|$ when G is generated with the β-model for $d_1, \ldots, d_n$.

The question is how similar are X and X_β.

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Barvinok and Hartigan (2012) proved:

For $|Y| \geq \delta n^2$,

$$(1 - \delta n^{-1/2} \log n)\, \mathbb{E}\, X_\beta \leq X \leq (1 + \delta n^{-1/2} \log n)\, \mathbb{E}\, X_\beta$$

with probability $1 - n^{-\Omega(n)}$.

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Isaev and McKay (2016) proved:

For any Y and any $\gamma > 0$,

$$\mathrm{Prob}\bigl(|X - \mathbb{E}\, X_\beta| \ge \gamma |Y|^{1/2}\bigr) \ge 1 - c e^{-2\gamma \min\{\gamma, n^{1/6} (\log n)^{-3}\}},$$

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The key to the improvement was a way to estimate n-dimensional complex integrals by casting them as complex martingales.

Counting regular graphs of high degree

The number of regular graphs can be written as a coefficient in a generating function:

$$\mathrm{RG}(n,d) = [x_1^d \cdots x_n^d] \prod_{j<k} (1 + x_j x_k).$$

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By applying Cauchy's Residue Theorem, we have

$$\mathrm{RG}(n,d) = \frac{1}{(2\pi i)^n} \oint \cdots \oint \frac{\prod_{j<k}(1 + x_j x_k)}{x_1^{d+1} \cdots x_n^{d+1}} \, dx_1 \cdots dx_n,$$

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Let's choose our contours to be circles:

$$x_j = re^{i\theta_j}, \text{ where } r = \sqrt{\frac{\lambda}{1-\lambda}}, \quad \lambda = \frac{d}{n-1}.$$

The case of high degree (continued)

Taking some stuff outside the integral:

$$\mathrm{RG}(n,d) = \frac{(1+r^2)^{\binom{n}{2}}}{(2\pi r^d)^n} I(n,d),$$

where

$$I(n,d) = \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} F(\theta_1,\ldots,\theta_n)\, d\theta_1 \cdots d\theta_n,$$

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$$F(\boldsymbol{\theta}) = \frac{\prod_{j<k}\bigl(1+\lambda(e^{i(\theta_j+\theta_k)}-1)\bigr)}{\exp\bigl(id\sum_j \theta_j\bigr)}.$$

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$|F(\boldsymbol{\theta})| \le 1$ always, which is achieved only at $(\theta_1,\ldots,\theta_n) = (0,\ldots,0)$ and $(\theta_1,\ldots,\theta_n) = (\pi,\ldots,\pi)$.

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Result: $\mathrm{RG}(n,d) =$

$$\sqrt{2}\,\bigl(2\pi n\lambda^{d+1}(1-\lambda)^{n-d}\bigr)^{-n/2} \exp\left(\frac{-1+10\lambda-10\lambda^2}{12\lambda(1-\lambda)} + o(1)\right)$$

if $d > n/\log n$. (McKay and Wormald, 1990)

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We noticed in 1990 that the expressions for low degree and high degree can be written in the same form. Recall the density $\lambda = d/(n-1)$.

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This false assumption gives an estimate

$$\widehat{\mathrm{RG}}(n,d) = \left(\lambda^{\lambda}(1-\lambda)^{1-\lambda}\right)^{\binom{n}{2}} \binom{n-1}{d}^{n}.$$

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Theorem. $\mathrm{RG}(n,d) \sim \sqrt{2}\, e^{1/4}\, \widehat{\mathrm{RG}}(n,d)$ for

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We conjectured that the theorem only requires $1 \le d \le n-2$.

Extended counting conjecture

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Theorem.
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Amount of irregularity

When $\bar{d} \approx cn$, the theorems we have mentioned require

$|d_j - \bar{d}| \le n^{1/2+\varepsilon}$ for all j (McKay and Wormald), or

$|d_j - \bar{d}| \le n^{3/5-\varepsilon}$ for all j, with c small enough (Liebenau and Wormald).

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Our aim is to achieve a similar variation of degrees but allow the average degree to be much smaller.

Why is the integral method difficult for smaller degree?

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In our problem, $\int_B |e^{\widehat{G}(\boldsymbol{\theta})}|$ is about $e^{n/\bar{d}}$ times larger than $\int_B e^{\widehat{G}(\boldsymbol{\theta})}$, so the effect of approximating $G(\boldsymbol{\theta})$ is catastrophic if $n/\bar{d} \to \infty$ quickly.

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A second problem is that $\int |F(\boldsymbol{\theta})|$ outside B is no longer small compared to $\int F(\boldsymbol{\theta})$ inside B, so we need a new method for that.

Excursion: cumulants of a random variable

Let Z be a random variable and let $\mathbb{E}$ denote expectation.

The central moments of Z are defined by

$$\mu_2(Z) = \mathbb{E}\,(Z - \mathbb{E}Z)^2,$$
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An alternative sequence of numbers is the cumulants:

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In general, the cumulants are defined by a formal series:

$$\mathbb{E}\,e^{tZ} = \sum_{j\ge 0} \frac{t^j}{j!}\,\mu_j(Z) = \exp\Bigl(\sum_{j\ge 0} \frac{t^j}{j!}\,\kappa_j(Z)\Bigr).$$

Cumulants (continued)

Now let $\boldsymbol{X} = (X_1, \ldots, X_n)$ be a vector of independent random variables and let $f(x_1, \ldots, x_n)$ be a complex-valued function.

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Isaev recently found a bound on the remainder when the cumulant series for $f(X_1, \ldots, X_n)$ is truncated:

$$\mathbb{E}\, e^{f(\boldsymbol{X})} = \exp\biggl(\sum_{j=0}^{s} \frac{1}{j!} \kappa_j(f(\boldsymbol{X})) + \text{Remainder} \biggr).$$

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The bound depends on generalised Lipshitz constants for f.

$$\begin{aligned} \varDelta_1 f &= \max \bigl| f(x_1, \ldots, x_j, \ldots, x_n) \\ &\qquad - f(x_1, \ldots, x_j', \ldots, x_n) \bigr| \\ \varDelta_2 f &= \max \bigl| f(x_1, \ldots, x_j, \ldots, x_k, \ldots, x_n) \\ &\qquad - f(x_1, \ldots, x_j', \ldots, x_k, \ldots, x_n) \\ &\qquad - f(x_1, \ldots, x_j, \ldots, x_k', \ldots, x_n) \\ &\qquad + f(x_1, \ldots, x_j', \ldots, x_k', \ldots, x_n) \bigr|, \ \text{etc.} \end{aligned}$$

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Now find a matrix S such that $S^T A S = I$ and change variables like $\boldsymbol{\theta} = S\boldsymbol{\phi}$ (a scaling and rotation in n-space), while choosing B to be a cube R aligned with the axes after the rotation.

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Now find a matrix S such that $S^T A S = I$ and change variables like $\boldsymbol{\theta} = S\boldsymbol{\phi}$ (a scaling and rotation in n-space), while choosing B to be a cube R aligned with the axes after the rotation.

This gives us an integral

$$C_1 \int_R e^{-\boldsymbol{\phi}^T\boldsymbol{\phi} + f(S\boldsymbol{\phi})},$$

which is $C_2\,\mathbb{E}e^{f(S\boldsymbol{X})}$ for $\boldsymbol{X}$ being a vector of independent truncated normal distributions and C_1, C_2 are some stuff we can figure out.

What do cumulants have to do with our problem?

Recall: We need to integrate $e^{G(\boldsymbol{\theta})}$ in a small region B.

The Taylor expansion for $G(\boldsymbol{\theta})$ looks like this:

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Now apply Isaev's cumulant series theorem to $e^{f(S\boldsymbol{X})}$.

The answer

The integral outside B is negligible (a difficult technical calculation outside the scope of this talk).

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If $\bar{d} \geq n^{\sigma}$ for some $\sigma > 0$, the number of graphs with degrees $d_1, \ldots, d_n$ is

$$\text{Stuff} \, \exp\Biggl(\sum_{j=0}^{2\lceil (1+p)/\sigma \rceil} \frac{1}{j!} \kappa_j(f(S\boldsymbol{X})) + O(n^{-p}) \Biggr),$$

for any p.

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for any p.

For $\bar{d} \approx cn$, we allow the degrees to vary by the same amount as Barvinok and Hartigan did.

For $\bar{d} = o(n)$, we only require that each degree lies in $[c_1\bar{d}, c_2\bar{d}]$ for some constants $0 < c_1 \leq c_2$.

The answer for regular graphs

For any J,

$$G(n,d) = \sqrt{2}\,\widehat{\mathrm{RG}}(n,d)\exp\biggl(\sum_{j=1}^{J}\frac{p_j(\Lambda)}{\Lambda^j n^{j-1}} + O(\Lambda^{-J-1}n^{-J})\biggr),$$

where $\Lambda = \lambda(1-\lambda)$ and p_j is a polynomial of degree j.

$$\begin{aligned}
p_1(x) &= \frac{1}{4}x,\\
p_2(x) &= -\frac{1}{4}x^2,\\
p_3(x) &= \frac{1}{24}(2-23x)x^2,\\
p_4(x) &= \frac{1}{24}(22-129x)x^3,\\
p_5(x) &= -\frac{1}{12}(3-115x+483x^2)x^3,\\
p_6(x) &= -\frac{1}{60}(375-6615x+22097x^2)x^4.
\end{aligned}$$

These are enough to re-prove the regular conjecture for $d \geq n^{1/7+\varepsilon}$.

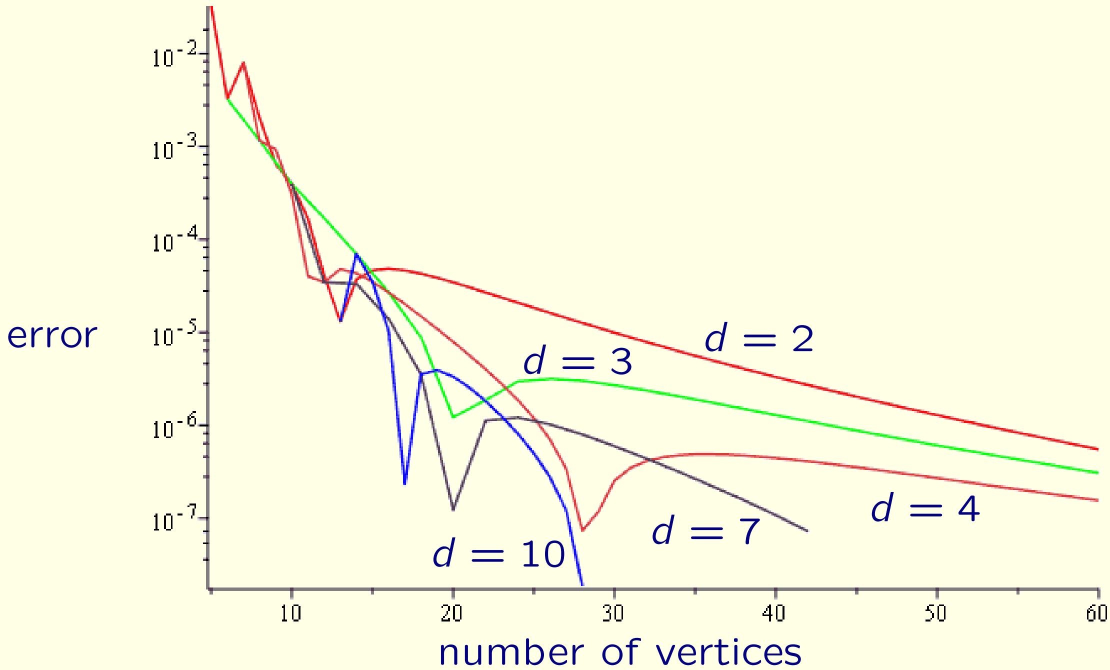
An example of the precision for regular graphs

$$G(n,d) = \sqrt{2}\,\widehat{\mathrm{RG}}(n,d)\exp\biggl(\sum_{j=1}^{J}\frac{p_j(\Lambda)}{\Lambda^j n^{j-1}} + O(\Lambda^{-J-1}n^{-J})\biggr),$$

Here is how it performs for RG(26, 12).

J	value	rel. err.
1	1.4258993×10^{77}	1.1×10^{-2}
2	1.4120471×10^{77}	1.0×10^{-3}
3	1.4107433×10^{77}	1.1×10^{-4}
4	1.4106066×10^{77}	1.6×10^{-5}
5	1.4105885×10^{77}	2.9×10^{-6}
6	1.4105853×10^{77}	6.5×10^{-7}
exact	1.4105844×10^{77}	

A new puzzle



The expansion seems to work for every d, even constant d, but we have no idea how to prove it.

Generalizing

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Think of that as
"all subgraphs of the complete graph K_n with a given degree sequence".

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The case where G is bipartite can also be done by similar methods, but we didn't do it yet.