Faithful subgraphs and Hamiltonian circles of infinite graphs

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Outline

Locally finite; Ray; Equivalent; End

G is a locally finite graph: all its vertices have finite degree.

A 1-way infinite path is called a ray of G, and the subrays of a ray are its tails.

Two rays of G are equivalent: for every finite set $S \subseteq V(G)$, there is a component of $G - S$ containing tails of both rays.
$R_1 \approx_G R_2$: R_1 and R_2 are equivalent in G.

The corresponding equivalence classes of rays are the ends of G.
$\Omega(G)$: the set of ends of G.

$C(S, \alpha)$: the unique component of $G - S$ that containing a ray (and a tail of every ray) in α.
$\Omega(S, \alpha)$: the set of all ends β with $C(S, \beta) = C(S, \alpha)$.

Freudenthal compactification; arc; circle; Hamiltonian

Freudenthal compactification $|G|$ (on $V(G) \cup E(G) \cup \Omega(G)$):

To built a topological space $|G|$ we associate each edge $uv \in E(G)$ with a homeomorphic image of the unit interval $[0, 1]$, where 0,1 map to u, v and different edges may only intersect at common endpoints. Basic open neighborhoods of points that are vertices or inner points of edges are defined in the usual way, that is, in the topology of the 1-complex. For an end α we let the basic neighborhood $\widehat{C}(S, \alpha) = C(S, \alpha) \cup \Omega(S, \alpha) \cup E(S, \alpha)$, where $S \subseteq V(G)$ is finite and $E(S, \alpha)$ is the set of all inner points of the edges between $C(S, \alpha)$ and S.

Arc: the image of a homeomorphic map of the unit interval $[0, 1]$ in $|G|$; Circle: the image of a homeomorphic map of the unit circle S^1 in $|G|$.
A circle of G is Hamiltonian if it meets every vertex (and then every end) of G.

Conjectures and results on Hamiltonian circles

Diestel launched the project of extending results on Hamiltonian cycles of finite graphs to Hamiltonian circles of infinite graphs. He specifically conjectured that the square of every 2-connected locally finite graph has a Hamiltonian circle, in order to obtain a unification of Fleischner's theorem [JCTB 1974]. This was confirmed by Georgakopoulos [Adv. Math. 2009].

Georgakopoulos then conjectured that the line graph of every 4-edge-connected graph has a Hamiltonian circle. Bruhn conjectured that Tutte's theorem on Hamiltonian cycles of 4-connected planar graphs can be extended to Hamiltonian circles. These conjectures are open but significant progress has been made by Lehner [JCTB 2014] on the former and by Bruhn and Yu [SIAMDM 2008] on the latter. Several other results in this area can be found in [Cui-Wang-Yu, JCTB 2009][Hamann-Lehner-Pott, Electronic J. Combin. 2016][Heuer, European J. Combin. 2015].

Faithful subgraph

We study a method for finding Hamiltonian circles in infinite graphs, which is closely related to the faithful subgraphs, the end degrees and the Hamiltonian curves. We apply this by extending several results on finite graphs to infinite ones. Specially we prove that the prism of every infinite 3-connected cubic graph has a Hamiltonian circle, extending the finite case by Paulraja.

Let G be an infinite locally finite graph. A subgraph F is faithful to G if
(1) every end of G contains a ray of F; and
(2) for each two rays R_1, R_2 of F, $R_1 \approx_F R_2$ if and only if $R_1 \approx_G R_2$.
If $F \leq G$, then for every finite set $S \in V(F)$, each component of $F - S$ is contained in a component of $G - S$. Thus the condition (2) can be replaced by 'for each two rays R_1, R_2 of F, $R_1 \approx_G R_2$ implies $R_1 \approx_F R_2$'.

End degree; Hamiltonian curve

The (vertex-)degree of an end $\alpha \in \Omega(G)$ is the maximum number of vertex-disjoint rays in α; and the edge-degree of α is the maximum number of edge-disjoint rays in α. We denote by $d(\alpha)$ the degree of α.

We define a curve of G as the image of a continuous map of the unit interval $[0, 1]$ in $|G|$. A curve is closed if $0, 1$ map to the same point; and is Hamiltonian if it is closed and meets every vertex of G exactly once. In other words, a Hamiltonian curve is the image of a continuous map of the unit circle S^1 in $|G|$ that meets every vertex of G exactly once. Thus a Hamiltonian curve may repeat at ends. Note that a Hamiltonian circle is a Hamiltonian curve but not vice versa.

Necessary and sufficient condition for Hamiltonian circles

Theorem

For an infinite locally finite graph G, the following three statements are equivalence:
(1) G has a Hamiltonian circle.
(2) G has a faithful spanning subgraph F with a Hamiltonian curve and for every $\alpha \in \Omega(F)$, $d(\alpha) = 2$.
(3) G has a faithful spanning subgraph F with a Hamiltonian curve and for every $\alpha \in \Omega(F)$, $d(\alpha) \leq 3$.

Theorem (Kündgen-Li-Thomassen European J. Combin. 2017)

Let G be an infinite locally finite graph. Then every Hamiltonian circle of G corresponds to a 2-factor F of G such that
(1) every finite cut intersects F a positive even number of edges,
(2) for each two distinct edges $e_1, e_2 \in F$, G has a finite edge-cut M such that $F \cap M = \{e_1, e_2\}$.
Conversely, if a 2-factor F satisfies (1)(2), then the closure of F is a Hamiltonian circle of G.

Lemmas to prove the main result

Lemma (for Faithful subgraphs)

Let G be an infinite locally finite graph, and let F be a faithful spanning subgraph of G. If F has a Hamiltonian circle, then G has a Hamiltonian circle.

Lemma (for end degrees)

If every end of G has degree at most 3, then every Hamiltonian curve of G is also a Hamiltonian circle.

Theorem (for Hamiltonian curves, Kündgen-Li-Thomassen)

An infinite locally finite graph G has a Hamiltonian curve if and only if every finite set $S \subseteq V(G)$ is contained in a cycle of G.

Properties of faithful subgraphs

Lemma

Suppose that G is infinite locally finite and $D \leq F \leq G$. Then each two of the following statements imply the third one: (1) D is faithful to F; (2) F is faithful to G; (3) D is faithful to G.

Lemma

Suppose that D is a faithful spanning connected subgraph of G. Then for any graph F with $D \leq F \leq G$, F is faithful to G and D is faithful to F.

Prisms of 3-connected cubic graphs

The prism of a graph G is the Cartesian product $G \square K_2$. Prisms over 3-connected planar graphs are examples of 4-polytopes. Rosenfeld and Barnette [DM 1973] showed that every 3-connected cubic planar graph has a Hamiltonian prism, under the assumption of the Four Color Conjecture, which is open at that time. Fleischner [Ann. Discrete Math. 1989] found a proof avoiding the Four Color Theorem. Eventually, Paulraja [JGT 1993] showed that planarity is inessential here.

Theorem (Paulraja, JGT 1993)

If G is a finite 3-connected cubic graph, then $G \square K_2$ is Hamiltonian.

As an application of our main result, we show that Paulraja's theorem can be extended to infinite graphs.

Theorem (cubic graphs)

If G is an infinite 3-connected cubic graph, then $G \square K_2$ has a Hamiltonian circle.

Prisms of squares of graphs and line graphs

Kaiser et al. also proved the following theorems concerning the prisms of squares of graphs and prisms of line graphs.

Theorem (Kaiser et al. JGT 2007)

If G is a finite connected graph, then $G^2 \square K_2$ is Hamiltonian.

Theorem (Kaiser et al. JGT 2007)

If G is a finite 2-connected line graph, then $G \square K_2$ is Hamiltonian.

We extend the two results to Hamiltonian circles.

Theorem (squares of graphs)

If G is an infinite locally finite connected graph, then $G^2 \square K_2$ has a Hamiltonian circle.

Theorem (line graphs)

Let G be an infinite locally finite graph. If $L(G)$ is 2-connected, then $L(G) \square K_2$ has a Hamiltonian circle.

Proof of the theorem of cubic graphs

A (finite or infinite) cactus is a subcubic graph G such that
(1) each block of G is a cycle or a K_2;
(2) every vertex is contained in one or two blocks
The cactus is even if each cycle is even.

Lemma (faithfulness of prisms)

Let G be an infinite locally finite graph, $F \leq G$, and D be a finite connected graph. If $F \leq^F G$, then $F \square D \leq^F G \square D$.

Lemma (Hamiltonian curves and end degrees of prisms)

Let G be an infinite even cactus. Then (1) $G \square K_2$ has a Hamiltonian curve; and (2) every end of $G \square K_2$ has degree 2.

Proof of the theorem of cubic graphs

Lemma (3-connected cubic to 2-connected bipartite)

Every infinite 3-connected cubic graph has a faithful spanning 2-connected bipartite subgraph.

Lemma (2-connected bipartite to cacti)

Every infinite 2-connected subcubic graph has a faithful spanning cactus.

Lemma (sequence of finite subgraphs to faithful subgraph)

Let G be an infinite locally finite connected graph and $\mathcal{F} = (F_i)_{i=1}^{\infty}$ be a sequence of finite connected subgraphs of G such that $F_i \leq F_{i+1}$ and $N_G(F_i) \subseteq V(F_{i+1})$. Set $F = \bigcup_{i=1}^{\infty} F_i$. Suppose that for every component H of $G - F_{i+1}$, $N_G(H)$ is connected in $F_{i+1} - F_i$. Then $F \leq^{FS} G$.

3-connected cubic to 2-connected bipartite

Lemma

Let G be a 3-connected cubic graph, and S be a finite subset of $V(G)$. Then there is a finite subset T of $V(G)$ such that $S \cup N_G(S) \subseteq T$ and for every component H of $G - T$, $N_G(H)$ is 3-connected in $G - S$.

Lemma

Let G be a subcubic graph, and $\mathcal{U}$ be a finite class of finite subsets of $V(G)$. Suppose that each $U \in \mathcal{U}$ is 3-connected in G. Then G has a finite subgraph F such that
(1) every subset $U \in \mathcal{U}$ is contained in one component of F; and
(2) every component of F is either an isolated vertex or a 2-connected bipartite graph.

Lemma

Let G be an infinite 3-connected cubic graph, and D be a finite subgraph of G each component of which is either an isolated vertex or a 2-connected bipartite graph. Then G has a finite 2-connected bipartite subgraph F with $D \leq F$.

3-connected cubic to 2-connected bipartite

Lemma

Let G be a connected graph and $\mathcal{F} = (F_i)_{i=1}^{\infty}$ be a sequence of finite connected subgraphs of G such that $F_i \leq F_{i+1}$. Set $F = \bigcup_{i=1}^{\infty} F_i$. Suppose that $F \leq^S G$, and for every component H of $G - F_{i+1}$, $N_F(H)$ is connected in $F_{i+1} - F_i$. Then $F \leq^{FS} G$.

Lemma

Let G be an infinite locally finite connected graph, and let $(\mathcal{D}_i)_{i=1}^{\infty}$ be a sequence of finite sets of finite connected subgraphs of G. Set $\mathcal{D} = \bigcup_{i=1}^{\infty} \mathcal{D}_i$ and $D = \bigcup \mathcal{D}$. Suppose that
(1) $V(G) = V(D)$ (i.e., $D \leq^S G$);
(2) if $j \geq i + 2$, then $\bigcup \mathcal{D}_i$ and $\bigcup \mathcal{D}_j$ are disjoint;
(3) $\bigcup \mathcal{D}_1$ is connected, and for every $D_{j+1} \in \mathcal{D}_{j+1}$, there is a unique $D_j \in \mathcal{D}_j$ such that $V(D_{j+1}) \cap V(D_j) \neq \emptyset$;
(4) for every component H of $G - \bigcup(\bigcup_{i=1}^{j} \mathcal{D}_i)$, there is a unique $D_{j+1} \in \mathcal{D}_{j+1}$ with $V(D_{j+1}) \cap V(H) \neq \emptyset$.
Then $D \leq^{FS} G$.

Thank you for attention!