X-minors and X-spanning subgraphs

Ghent Graph Theory Workshop *on Structure and Algorithms*

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Barnette's Result

Barnette's Result (1966)

Every 3-connected planar graph G has a spanning tree of maximum degree at most 3.

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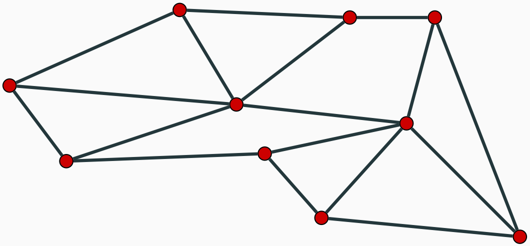

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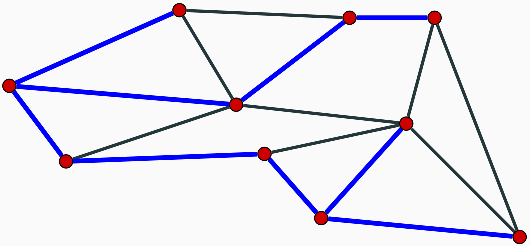

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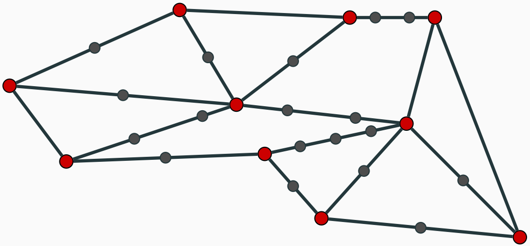

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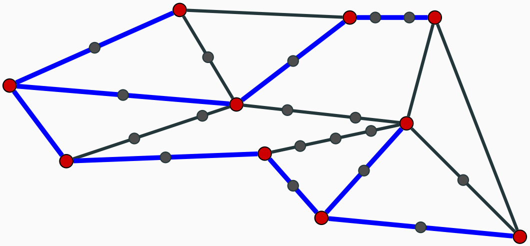

X-connectivity

Let G graph, $X \subseteq V(G)$:

X-separator:

- $S \subseteq V(G)$ is an X-separator
  $:\Leftrightarrow$ at least two components of $G - S$ contain a vertex from X.

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X-spanning:

- A subgraph H of G is X-spanning
  :$\Leftrightarrow$ $X \subseteq V(H)$.

Spanning subgraph results

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Is the following true?

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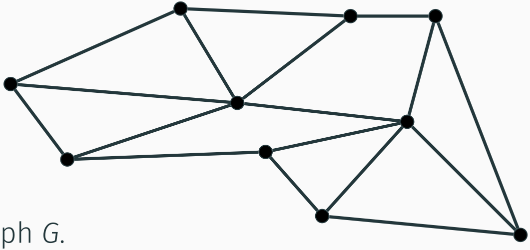
Concept?

??

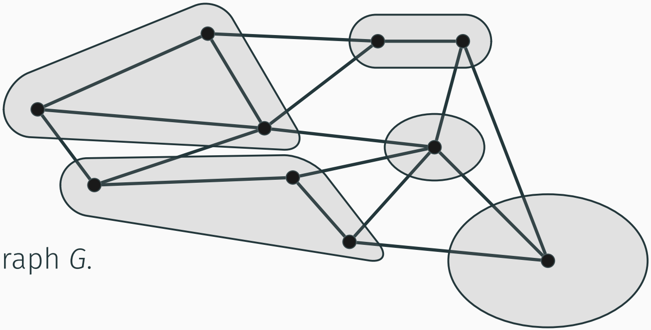
How to obtain X-spanning subgraph results?

Minors



Graph G.

Minors



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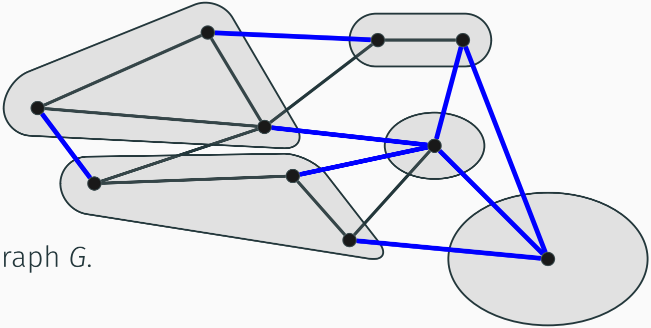

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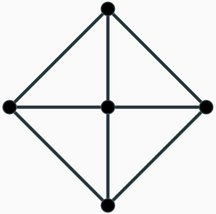

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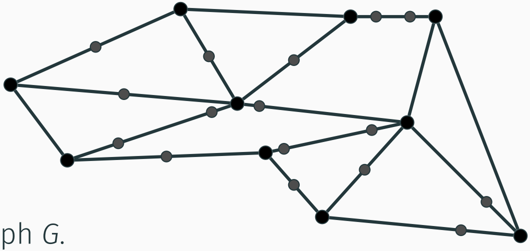

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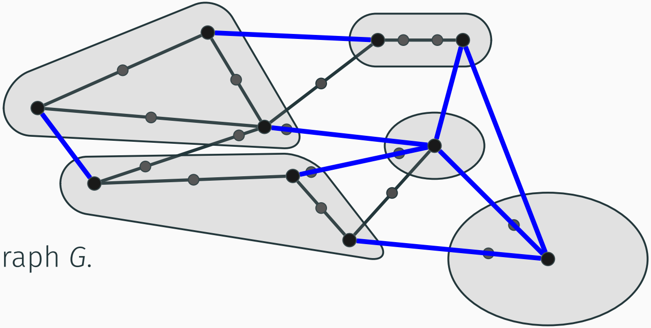

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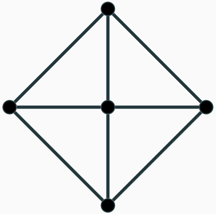

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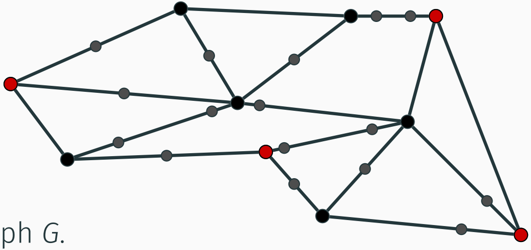

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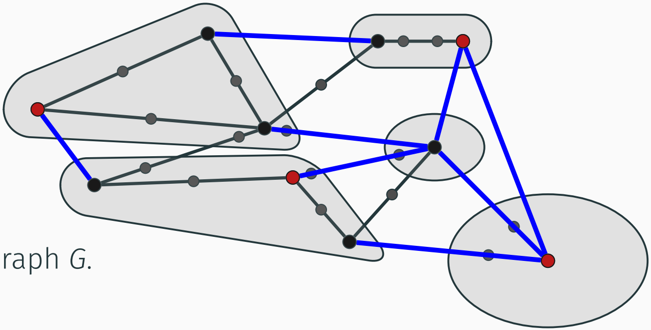

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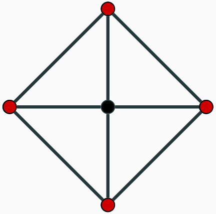

Definition

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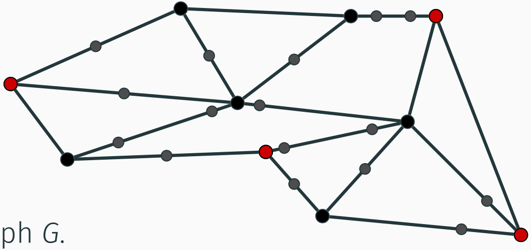
X-Minor:

- Each bag contains at most one vertex of X.
- Each $x \in X$ is contained in some bag.

Topological minors



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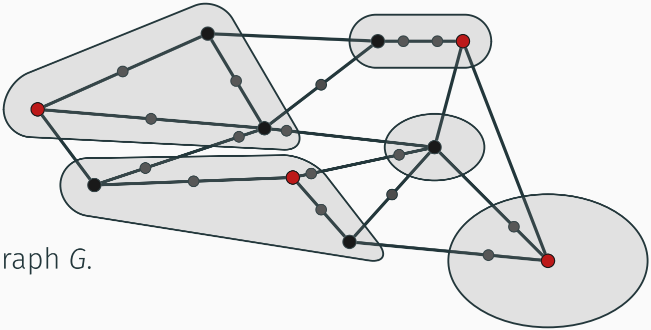

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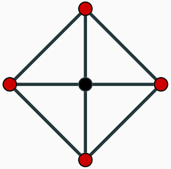

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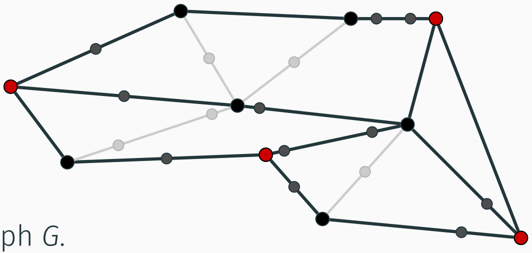

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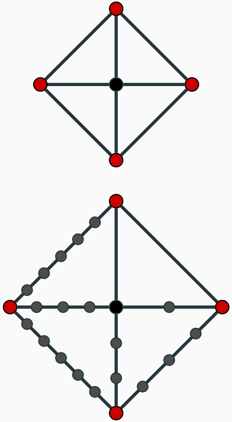

Theorem on topological X-minors

Topological X-Minor M^*:

- M^* is X-minor.
- A subdivision of M^* is subgraph of G such that X-vertices corresponds.

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Theorem (on topological X-minors)

1. *If $k \in \{1, 2, 3\}$, G is a graph, and $X \subseteq V(G)$ is k-connected in G, then G has a k-connected topological X-minor.*
2. ...

More X-spanning results

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- T is X-spanning tree in G of maximum degree at most 3.

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If G is a planar graph, $X \subseteq V(G)$, X is 3-connected in G, then G has an X-spanning tree of maximum degree at most 3.

Gao (1995)

Every 3-connected planar graph G contains a 2-connected spanning subgraph of maximum degree at most 6.

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Ota, Ozeki (2009)

Let $t \geq 4$ be an even integer. Every 3-connected graph G without $K_{3,t}$-minor has a spanning tree of maximum degree at most $(t-1)$.

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X-spanning version of Tutte's Theorem

If G is a planar graph, $X \subseteq V(G)$, X is highly connected in G, then G has an X-spanning cycle ???

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Theorem (on topological X-minors)

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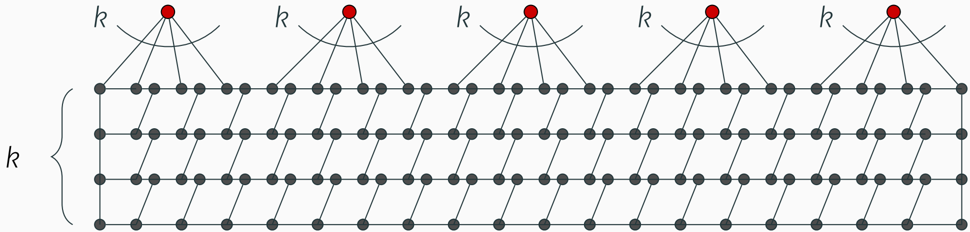
Problem

Arbitrary integer $k \geq 4$, planar graph F_k:



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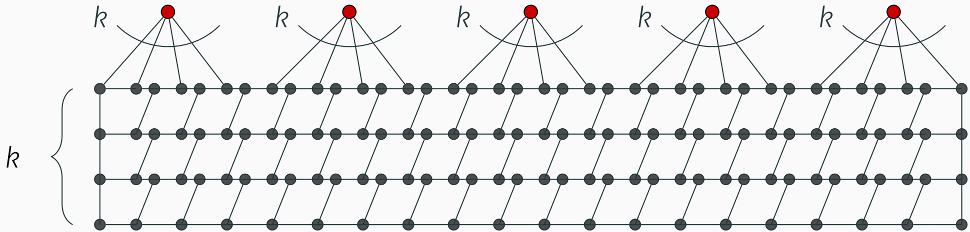

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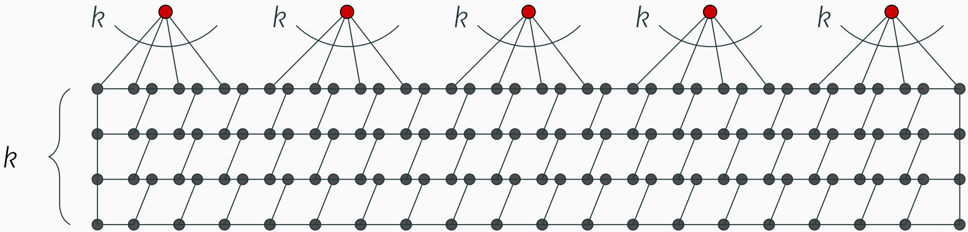

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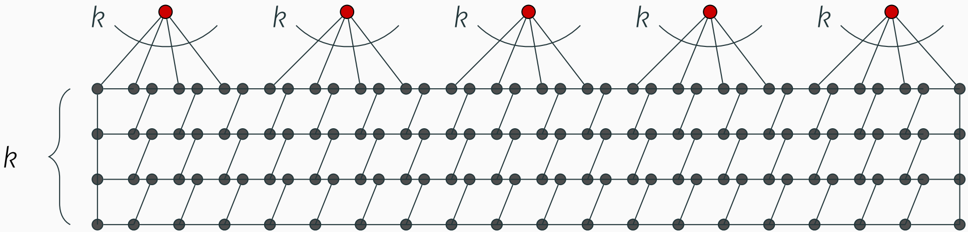

- X is k-connected in G.
- If M^* is topological X-minor, M^* 4-connected, then $\delta(M^*) \geq 4$.
- Then $V(M^*) = X$!

Theorem on topological X-minors

Theorem 1 (on topological X-minors)

1. *If $k \in \{1, 2, 3\}$, G is a graph, and $X \subseteq V(G)$ is k-connected in G, then G has a k-connected topological X-minor.*
2. *For an arbitrary integer k, there are infinitely many planar graphs G and $X \subseteq V(G)$ such that X is k-connected in G and G has no 4-connected topological X-minor.*

??

Is there a 4-connected ~~topological~~ X-minor?

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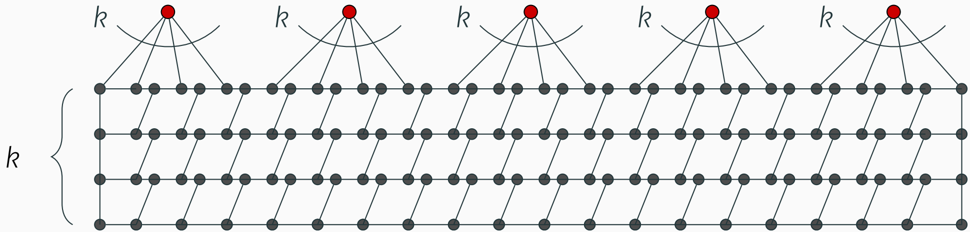

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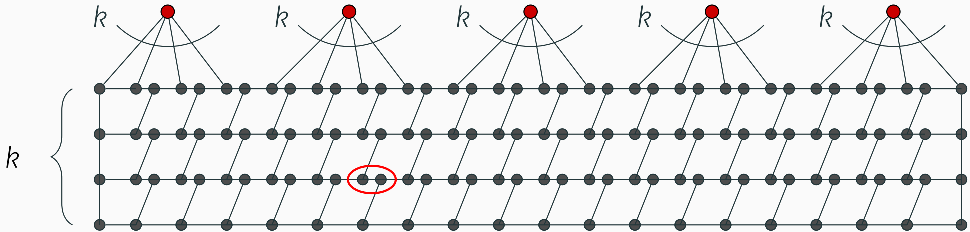

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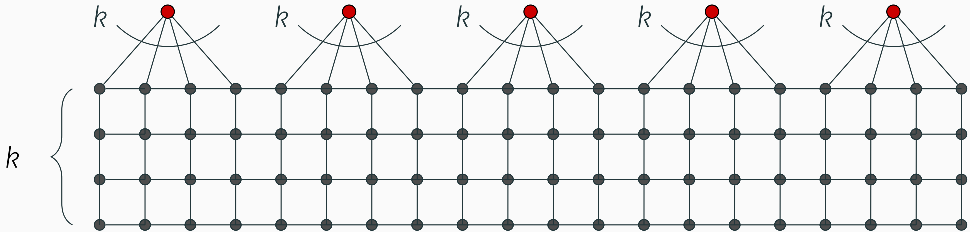

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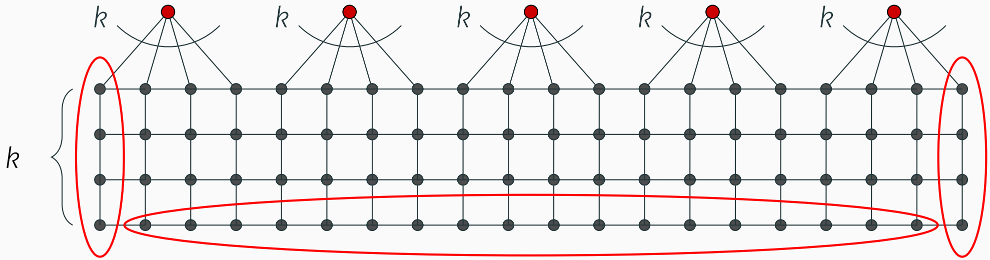

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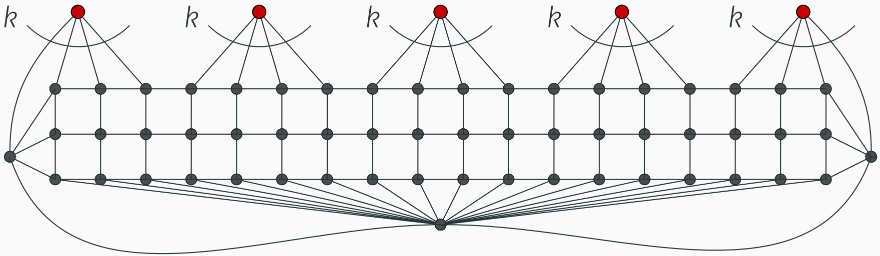

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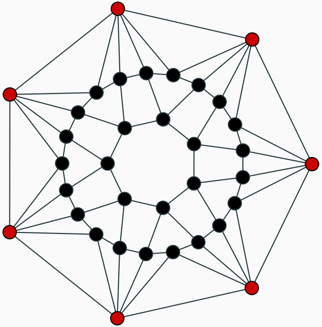

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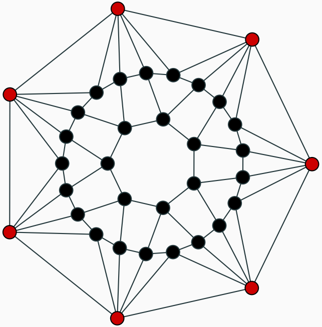

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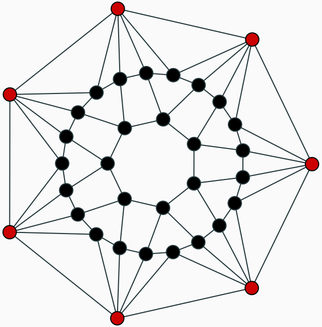

- X is 6-connected in G,
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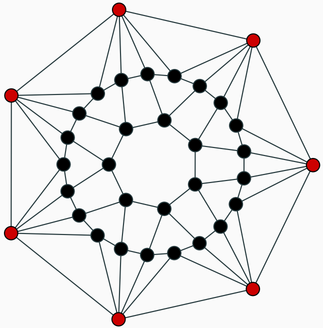

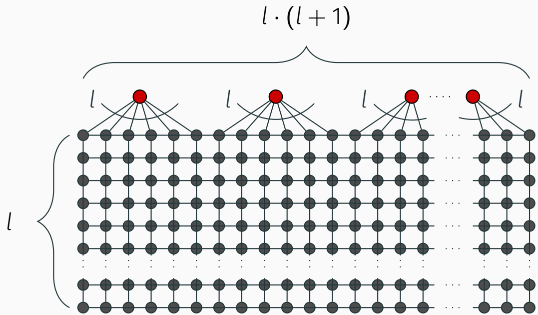

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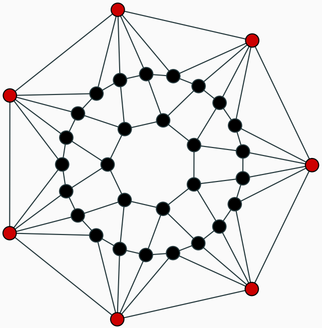

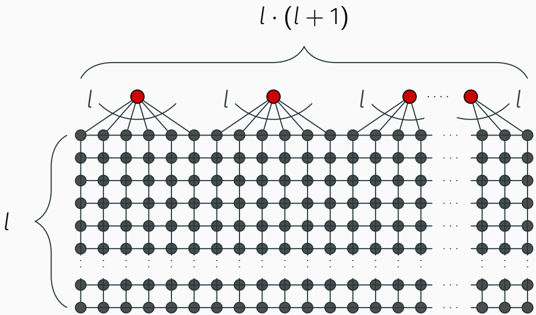

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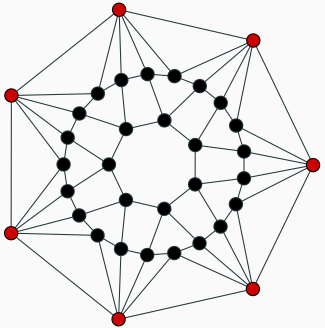

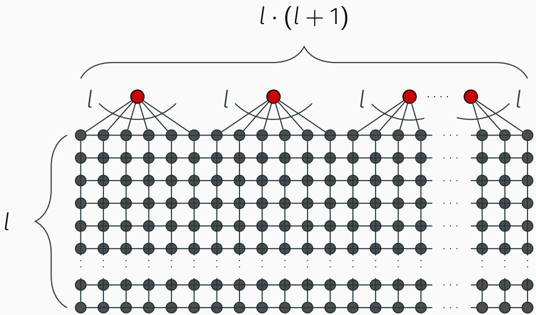

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- X is l-connected in G,
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Theorem on X-minors

Theorem 2 (on X-minors)

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2. *There are infinitely many planar graphs G and $X \subseteq V(G)$ such that X is 6-connected in G and G has no 5-connected X-minor.*
3. *For an arbitrary integer k, there are infinitely many planar graphs G and $X \subseteq V(G)$ such that X is k-connected in G and G has no 6-connected X-minor.*

Application

Ellingham (1996)

If G is a 4-connected graph embedded into a closed surface of Euler characteristic $\Sigma < 0$. Then there is a function $f(\,\cdot\,)$, such that G has a spanning tree of maximum degree at most $f(\Sigma)$.

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X-spanning version of Ellingham's Theorem

If G is a graph of Euler characteristic Σ, $X \subseteq V(G)$, X is 4-connected in G, then G has an X-spanning tree of maximum degree at most $f(\Sigma) + 1$.

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Bedankt voor uw aandacht