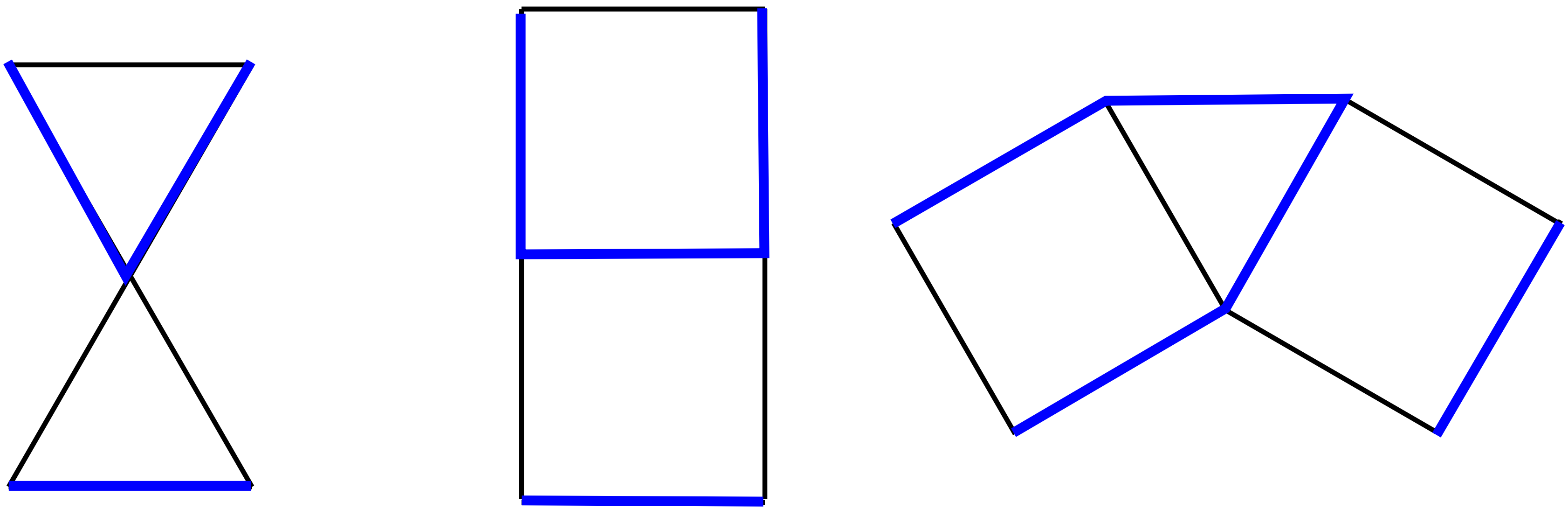
4-Connected Polyhedra have a Linear Number of Hamiltonian Cycles



Gunnar Brinkmann, Nico Van Cleemput

Concerning **hamiltonicity** for

plane triangulations and **polyhedra**

the **same results** seem to hold –

though they can have **much fewer edges**.

(Ratio: $\frac{3|V|-6}{2|V|}$)

- Whitney (1931): 4-connected plane triangulations are hamiltonian

- Tutte (1956): 4-connected polyhedra are hamiltonian

(25 years)

- Jackson, Yu (2002): plane triangulations with at most three 3-cuts are hamiltonian

- B., Zamfirescu (2019): polyhedra with at most three 3-cuts are hamiltonian

(17 years)

- plane triangulations with six 3-cuts can be non-hamiltonian

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- Hakimi, Schmeichel, Thomassen (1979): 4-connected planar triangulations have at least $|V|/\log|V|$ hamiltonian cycles.

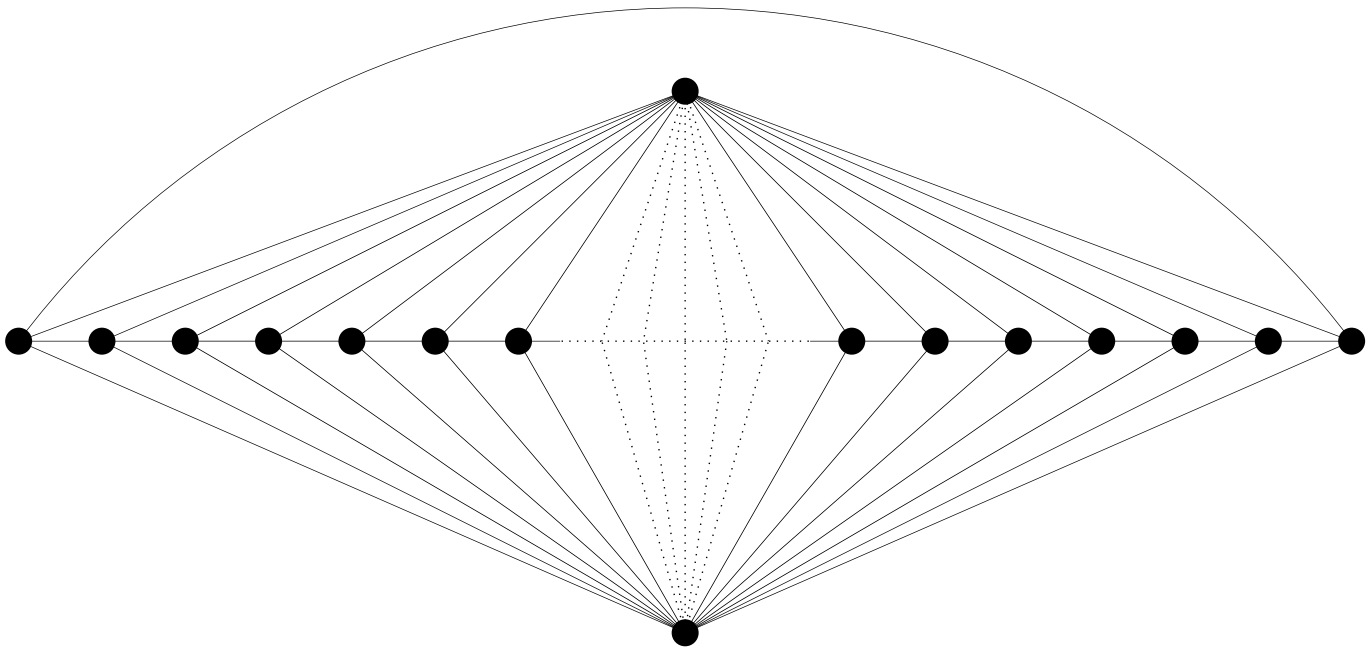
  (improved to $\frac{12}{5}(|V|-2)$ (2018), B., Souffriau, Van Cleemput)

- From a result of Thomassen (1983): 4-connected polyhedra have at least 6 hamiltonian cycles.

already 40 years ago. . .

(Alahmadi, Aldred, Thomassen 2019: 5-connected triangulations have an exponential number of hamiltonian cycles)

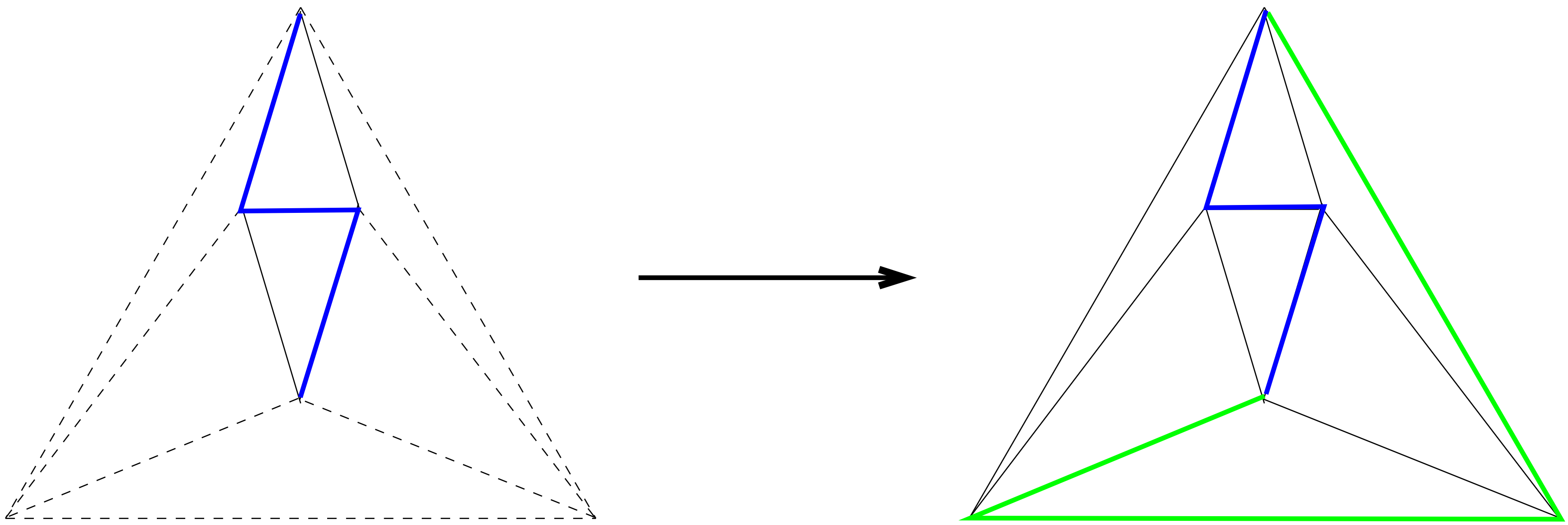
Only trivial lower bounds are known, but computations suggest that for $|V| \geq 18$ this is the 4 connected **polyhedron** with the smallest number of hamiltonian cycles:



$2|V|^2 - 12|V| + 16$ hamiltonian cycles

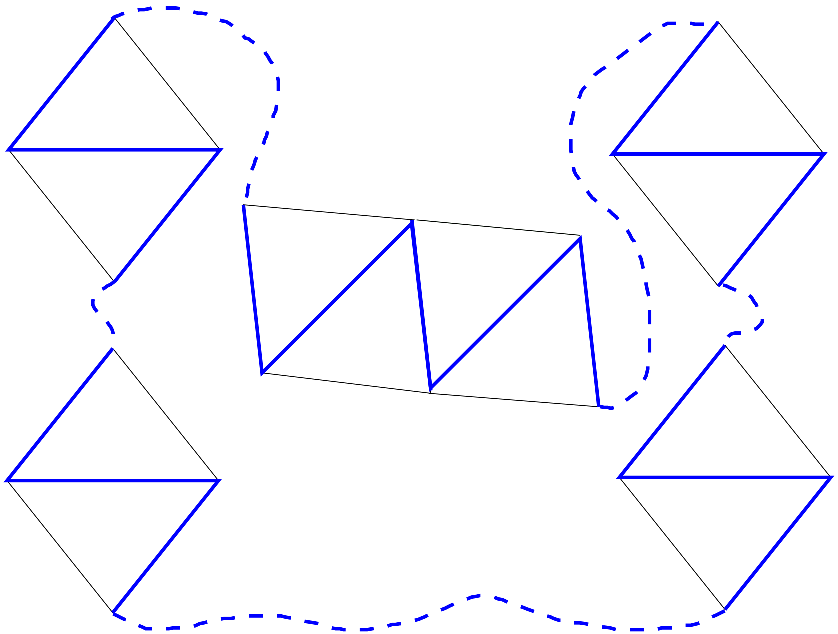
Hakimi, Schmeichel, Thomassen (1979)
with result of Whitney (1931):

Each **zigzag** in a triangle-pair in a
4-connected triangulation can be extended
to a hamiltonian cycle.



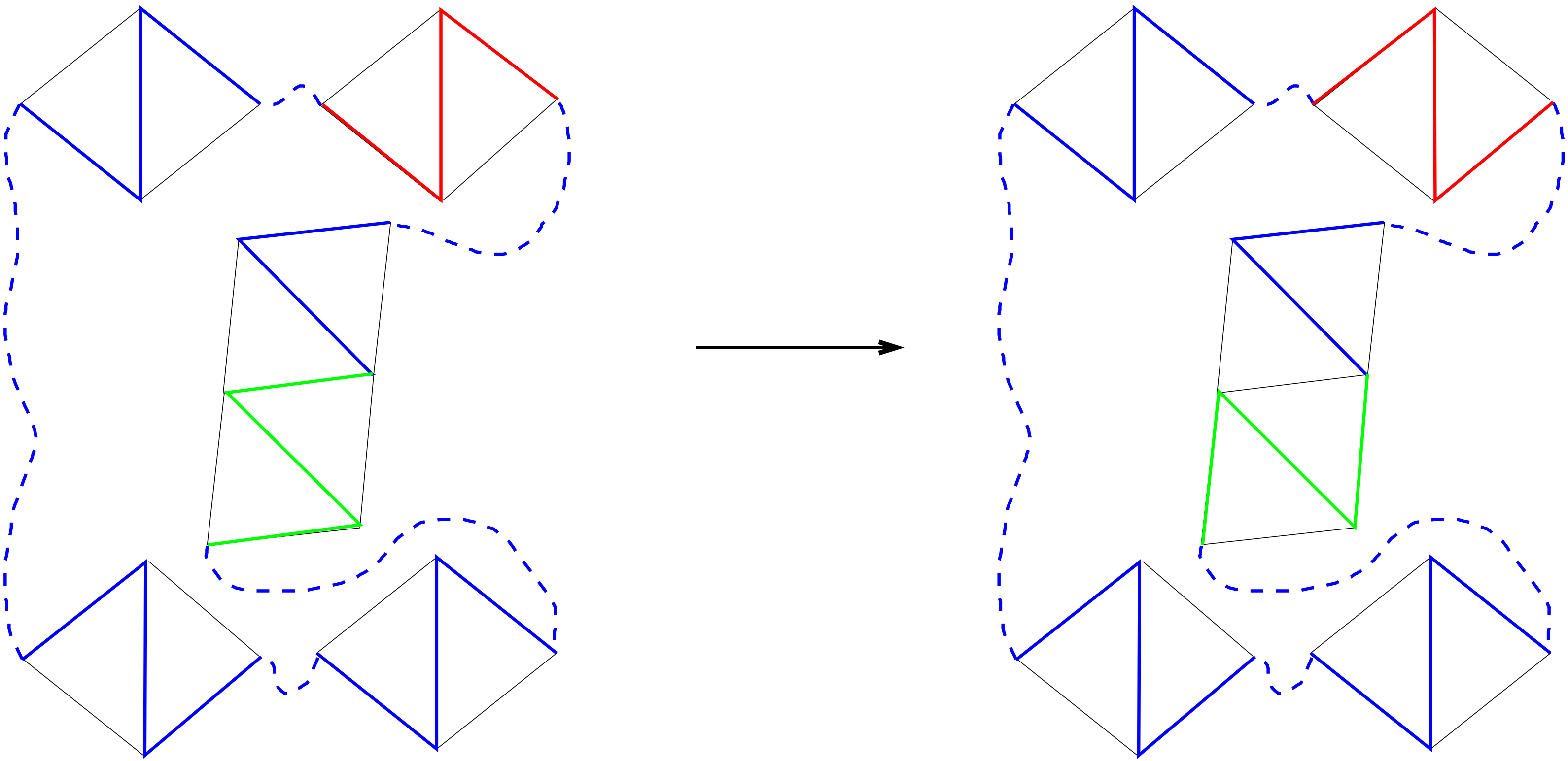
There is a linear number of such zigzags.

Problem: a single hamiltonian cycle can contain a linear number of these zigzags. . .



. . . giving in total a constant number of hamiltonian cycles.

A hamiltonian cycle with k disjoint zigzags guarantees 2^k hamiltonian cycles by *"switching"*.



This explains the $\ldots / \log |V|$ in the formula.

The main contribution of the 2018-paper: *counting differently* via *counting bases*:

Definition:

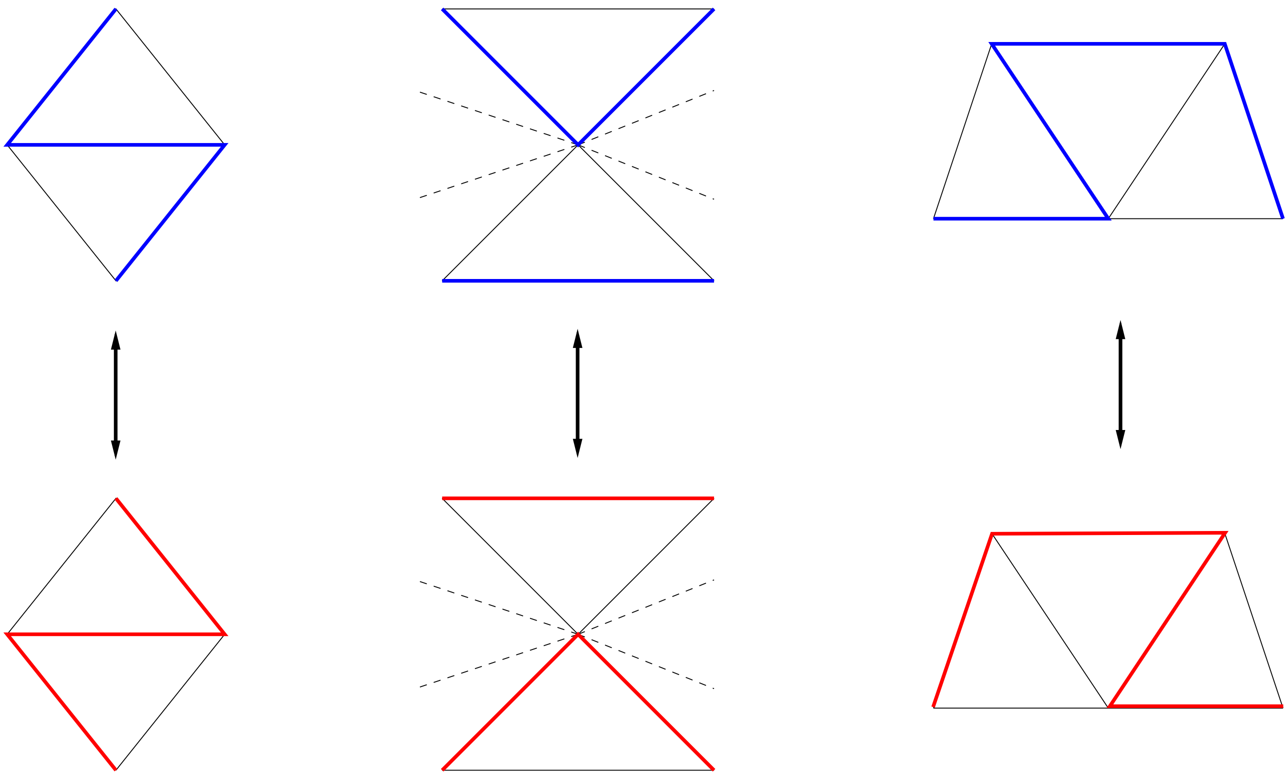
Let G be a graph and let $\mathcal{C}$ be a collection of hamiltonian cycles of G. The pair $(\mathcal{S}, r)$, where $\mathcal{S} \subset 2^{E(G)}$ and r is a function $r : \mathcal{S} \to 2^{E(G)}$, is called a *counting base* for G and $\mathcal{C}$ if the pair $(\mathcal{S}, r)$ has the following properties:

(i) for all $S \in \mathcal{S}$, there is a hamiltonian cycle $C \in \mathcal{C}$ saturating S.

(ii) for all $S \in \mathcal{S}$, $r(S) \subseteq E(G)$ (not necessarily in $\mathcal{S}$) so that $S \not\subset r(S)$ and for each hamiltonian cycle $C \in \mathcal{C}$ saturating S we have that $z(C, S) = (C \setminus S) \cup r(S)$ is a hamiltonian cycle in $\mathcal{C}$.

(iii) for all $S_1 \neq S_2$, $S_1, S_2 \in \mathcal{S}$ and C saturating S_1 and S_2, we have that $z(C, S_1) \neq z(C, S_2)$.

Informally: A *switching subgraph* is a subgraph that can be extended to a hamiltonian cycle and can be switched.



Very informally:

The counting base lemma:

If one has a set S of switching subgraphs, so that each switching subgraph overlaps with at most c others, then there are at least $|S|/c$ hamiltonian cycles.

Two big problems for polyhedra:

(a) The subgraphs must be extendable to hamiltonian cycles in polyhedra – not just in triangulations.

(b) Unlike triangulations, polyhedra can locally look very differently – there might e.g. be no triangle pairs.

Some polyhedra do not have a single of the switching subgraphs we have seen so far.

The key for solving (a):

Lemma: (Jackson, Yu, 2002)

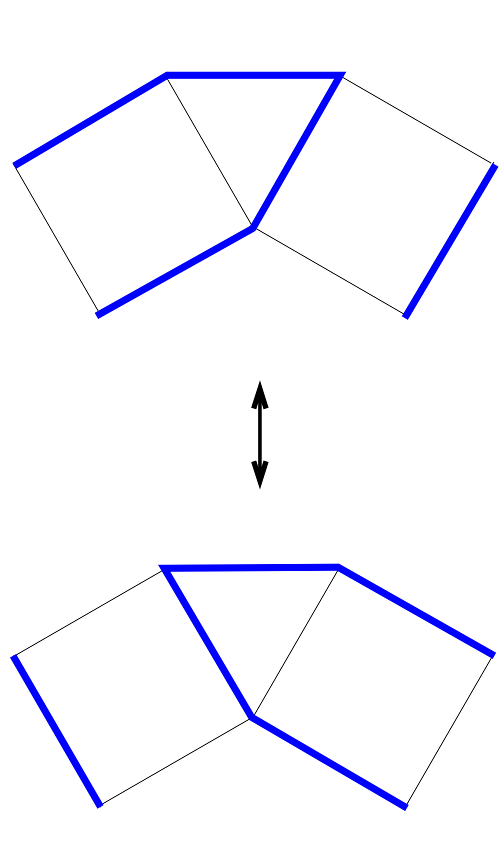
Let (G, F) be a circuit graph, r, z be vertices of G and $e \in E(F)$. Then G contains an F-Tutte cycle X through e, r and z.

Circuit graph: G plane, 2-connected, F facial cycle, for each 2-cut each component contains elements from F

$\mathbf{F}$-Tutte cycle: cycle C, so that bridges contain at most 3 endpoints on C and at most 2 if it contains an edge of F.

With Jackson/Yu:
In a 4-connected polyhedron each of the following subgraphs can be extended to a hamiltonian cycle,

if it is present in the polyhedron. . .



Unfortunately

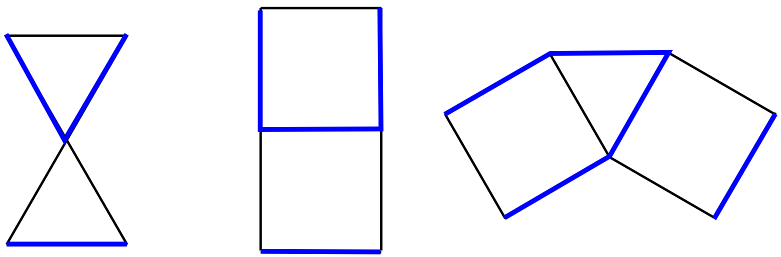
- for each of those switching subgraphs there are 4-connected polyhedra not containing it

- for each pair of those switching subgraphs there are 4-connected polyhedra containing only a small constant number of them

but

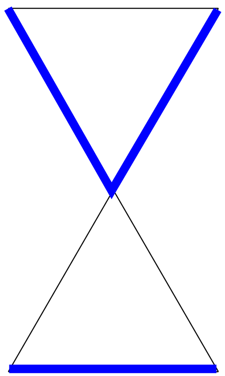
Theorem

Each 4-connected polyhedron has a linear number of those three switching subgraphs.



So with the counting base lemma:
4-connected polyhedra have at least a linear number of hamiltonian cycles.

Let f_i denote the faces of size i.

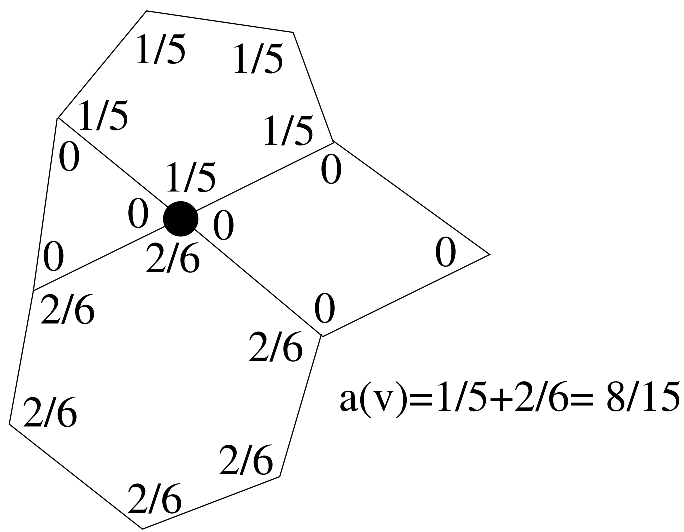


Lemma

- A polyhedron has at least $3f_3 - |V|$ hour-glasses.
- $f_3 \geq 8 + \sum_{i>4} (i-4) f_i$

Assign the value 0 to angles of triangles and quadrangles and value $\frac{i-4}{i}$ to each angle of an i-gon with $i > 4$.

Define $a(v)$ as the sum of all angle values around v.



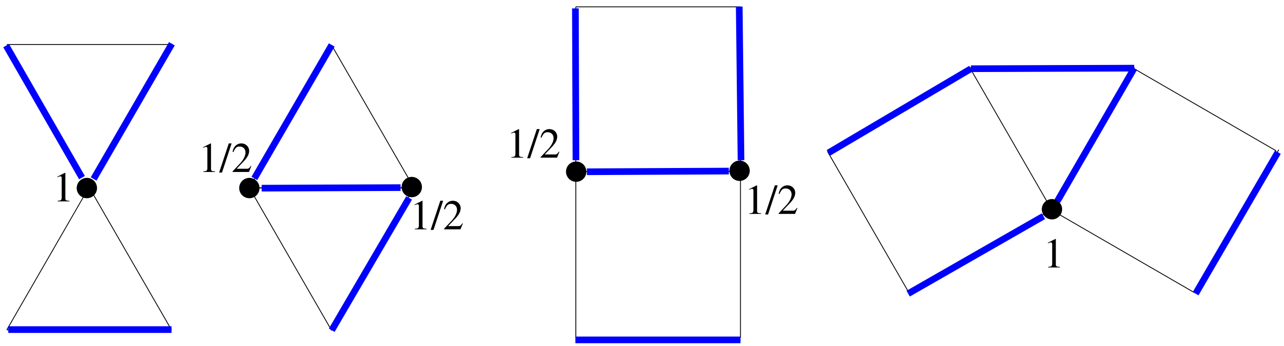
$$\sum_{v \in V} a(v) = \sum_{i>4} (i-4) f_i$$

As hourglasses are switching subgraphs:

With $\mathcal{S}_w$ the set of switching subgraphs this gives

$$|\mathcal{S}_w| \geq 24 + 3\sum_{v\in V} a(v) - |V|$$

Furthermore assign the following weights $w'(v)$ to vertices in switching subgraphs:



With $w(v)$ the sum of all $w'(v)$ we have: $\sum_{v \in V} w(v) = |\mathcal{S}_w|$

Lemma

Let $G = (V, E)$ be a plane graph with minimum degree 4. Then for each $v \in V$ we have

$$a(v) + w(v) \geq \frac{2}{5}$$

so

$$\sum_{v \in V} a(v) + |\mathcal{S}_w| \geq \frac{2}{5}|V|$$

Lemma:

For 4-connected polyhedra we have

$|\mathcal{S}_w| \geq \frac{1}{20}|V| + 6.$

So: 4-connected polyhedra have at least a linear number of hamiltonian cycles.

Proof: Set $a(V) = \sum_{v \in V} a(v)$.

We have two equations:

$$|\mathcal{S}_w| \geq 24 + 3a(V) - |V|$$

$$|\mathcal{S}_w| \geq \frac{2}{5}|V| - a(V)$$

compute intersection

Lemma:

Polyhedra $G = (V, E)$ with at most one 3-cut and for some $c > 0$ at least $(2 + \frac{2}{33} + c)|V|$ edges have at least a linear number of hamiltonian cycles.

Thank you for your attention!