Minimum leaf number of cubic graphs

Gábor Wiener

Department of Computer Science and Information Theory
Budapest University of Technology and Economics

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Joint work with Jan Goedgebeur, Kenta Ozeki, and Nico Van Cleemput

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Connectivity 2:

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E.g. the (unique) smallest 2-connected nonhamiltonian cubic planar bipartite graph has order 26.

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Proposition

$\mu(G) + 1 \leq \mathrm{ml}(G) \leq 2\mu(G)$.

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2-connected examples with $\mu(G) = \frac{n}{14}$ (G.-O.-V.-W., 2016)

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Theorem (Salamon-W., 2008)

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Theorem (Boyd-Sitters-van der Ster-Stougie, 2014)

If G is a 2-connected cubic multigraph of order n, then $\mathrm{ml}(G) \leq \frac{n}{6} + \frac{2}{3}$.

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Theorem (G.-O.-V.-W., 2016)

If G is a 2-connected cubic graph of order n, then $\mathrm{ml}(G) \le \frac{19n}{117} \approx \frac{n}{6.157}$.

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If G is a (2-connected) bipartite cubic graph of order n, then $\mathrm{ml}(G) \leq \lceil \frac{n}{20} \rceil + 1$.

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Question

If G is a 3-connected cubic planar graph of order n, then $\mathrm{ml}(G) \leq \lceil \frac{n}{72} + \frac{1}{2} \rceil$?

Thank you!