Smallest snarks with oddness 4

Martin Škoviera

Comenius University, Bratislava

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joint work with Jan Goedgebeur and Edita Máčajová

Snarks

Definition

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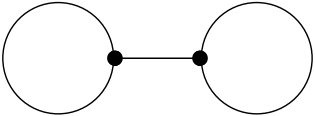
– trivially true for 3-edge-colourable graphs
– open for snarks
– verified for graphs "close" to 3-edge-colourable graphs

Small snarks

- smallest snark



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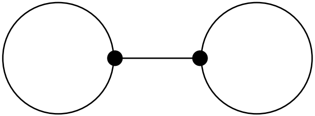
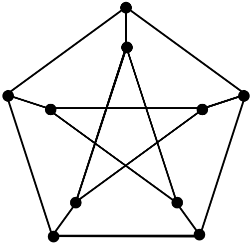


- smallest 2-connected snark



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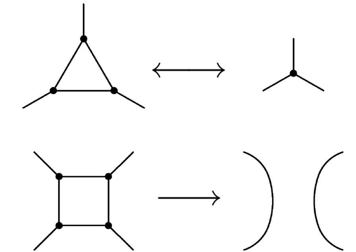

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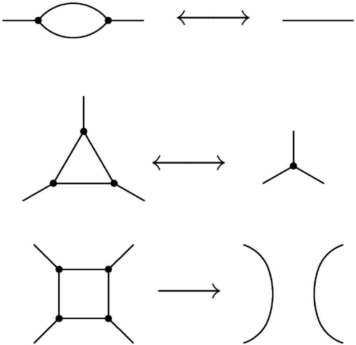

Similar simplifications for cycle-separating edge-cuts of size ≤ 3

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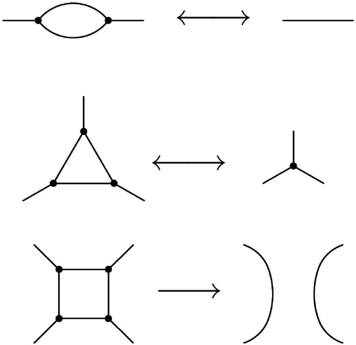


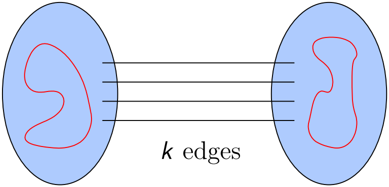
Similar simplifications for cycle-separating edge-cuts of size ≤ 3

$\Longrightarrow$ 'nontrivial' usually means

- cyclically 4-edge-connected, and
- girth > 4

Cyclic connectivity

Cyclic connectivity is the smallest number of edges whose removal leaves at least two components containing cycles.



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Theorem

The following statements are equivalent for a bridgeless cubic graph G:

- G *is* 3*-edge-colourable*
- $\omega(G) = 0$
- $\rho(G) = 0$

Parity Lemma

Lemma (Parity lemma)

Let M be a k-pole edge-coloured with three colours. Let k_i be the number of dangling edges coloured with colour i. Then

$$k_1 \equiv k_2 \equiv k_3 \equiv k \pmod 2.$$

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Corollary

For every snark G the following hold:

- $\rho(G) \geq 2$
- $\rho(G) = 2 \iff \omega(G) = 2$

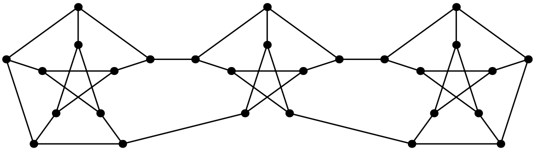
Small snarks with large oddness

Theorem

For every snark G there exists a snark G' of order not exceeding that of G such that $\omega(G') \geq \omega(G)$ and the girth of G' is at least 5.

Corollary

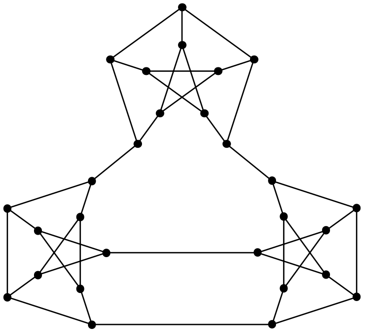
Every snark with oddness $\geq \omega$ and minimum order has girth at least 5.

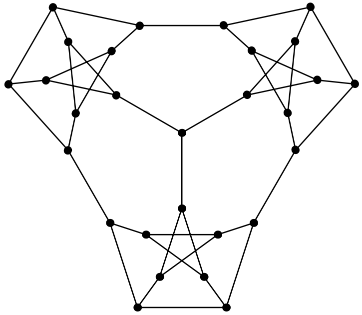


Small snarks with $\omega \geq 4$

Theorem

The smallest order of a snark with $\omega \geq 4$ is 28. There are exactly three such snarks: two with connectivity 2 and one with connectivity 3.





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Brinkmann, Goedgebeur, Hägglund & Markström (2013)
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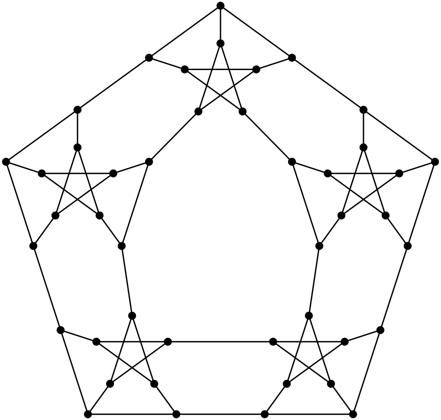
Theorem (Brinkmann et al. (2013))

All cyclically 4-edge-connected snarks up to 36 *vertices have* $\omega = 2$.

Snark of order 44 and $\omega = 4$



Constructed by Lukot'ka, Máčajová, Mazák & S. (2015)

Main result

Theorem

- *The smallest number of vertices of a snark with cyclic connectivity* 4 *and* $\omega \geq 4$ *is* 44.
- *There are exactly* 31 *such snarks, all with* $\omega = 4$.

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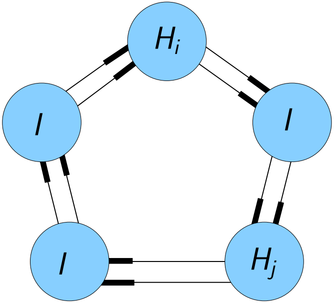
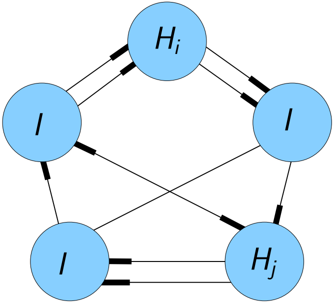


(a) Group 1a: 3 graphs



(b) Group 1b: 4 graphs

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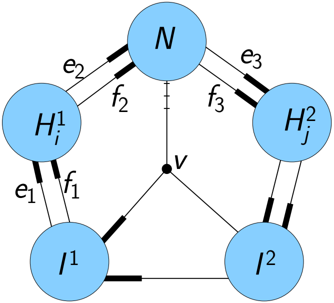
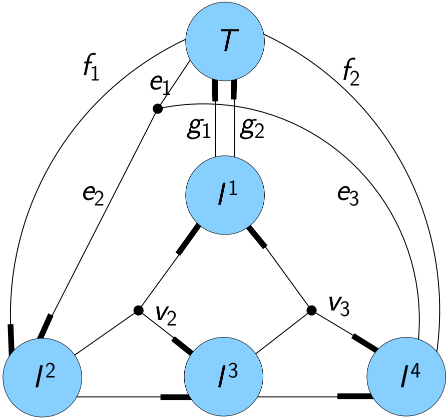


(c) Group 4: 4 graphs



(d) Group 6: 2 graphs

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- *There are exactly* 31 *such snarks, all with* $\omega = 4$.

The proof has two main ingredients:

- Analysis of 3-edge-colourings conflicting on a cycle-separating 4-edge-cuts.
- A "closure theorem" for cyclically 4-edge-connected graphs of Andresen, Fleischner & Jackson (1988)

Closure Theorem

Theorem (Andersen, Fleischner & Jackson, 1988)

Let H be a cyclically 4-edge-connected cubic graph with a cycle-separating 4-edge-cut S. Then each component of $H - S$ can be extended to a cyclically 4-edge-connected graph by adding two adjacent vertices.

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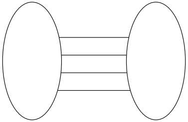

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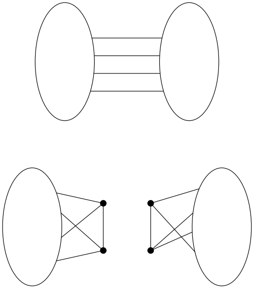

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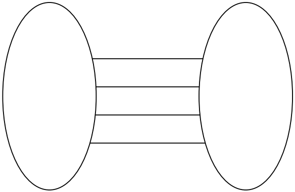

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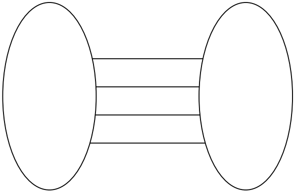

1	**1**	**1**	**1**
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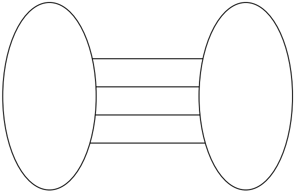

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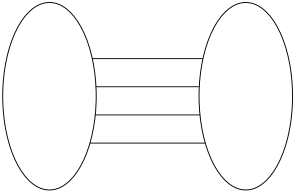

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- If both sides are colourable, then both have two types of colourings $\Rightarrow \rho(G) = 2 \Rightarrow \omega(G) = 2$ ... a contradiction.

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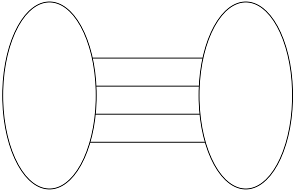

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- $\Rightarrow$ At least one part is not 3-edge-colourable.

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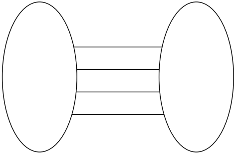

We prove that each side can be extended to a snark by adding either

- two adjacent vertices, or
- two isolated edges

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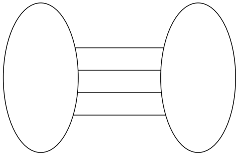

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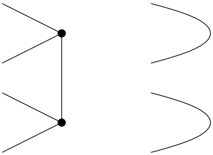

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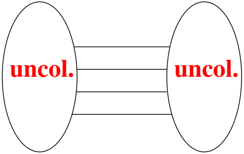

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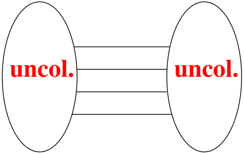

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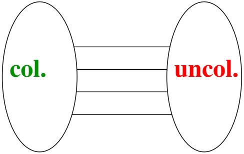

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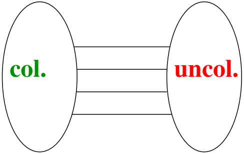

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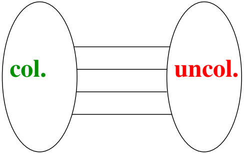

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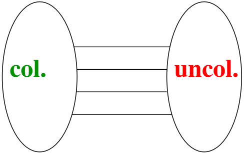

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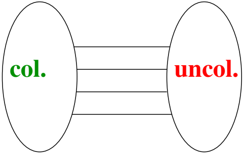

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  - We extend G_1 to a snark $\tilde{G}_1$ (the manner may be forced!)
  - By minimality, $\tilde{G}_1$ is also cyclically 4-edge-connected.

End of proof

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- We have computationally verified all possible pairs of snarks and checked that none of them has $\omega \geq 4$.

Final remarks

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To prove a similar result for nontrivial snarks (rather than just those with cyclic connectivity $= 4$) we need to exclude the existence of cyclically 5-edge-connected snarks of orders $38 - 42$ with $\omega \geq 4$.

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The existence of such snarks is unlikely.

Thank you!