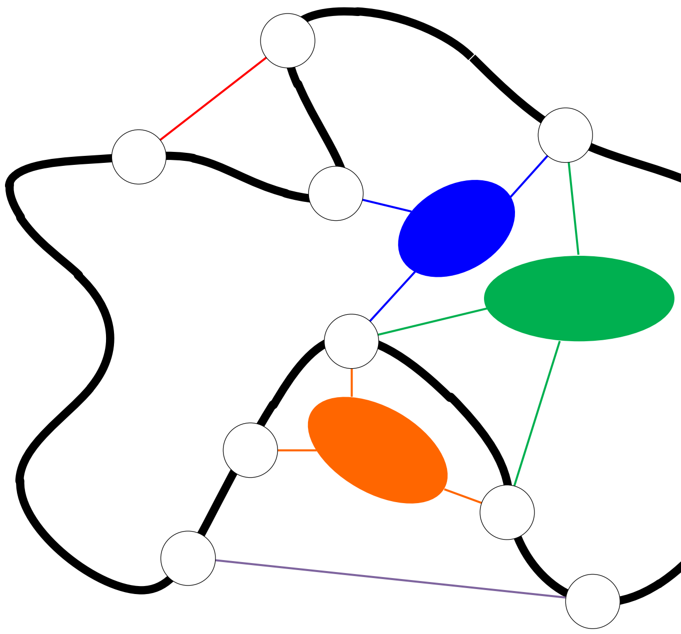
# Longer Cycles in Essentially 4-Connected Planar Graphs



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joint work with Igor Fabrici, Jochen Harant and Samuel Mohr

# Circumference

- $G$ := polyhedral graph, i.e. planar and 3-connected
- *circ(G)* := length of a longest cycle of $G$ (circumference)
  → If circ($G$) = $n := |V|$, $G$ has a Hamiltonian cycle.

What is the circumference of polyhedral graphs?

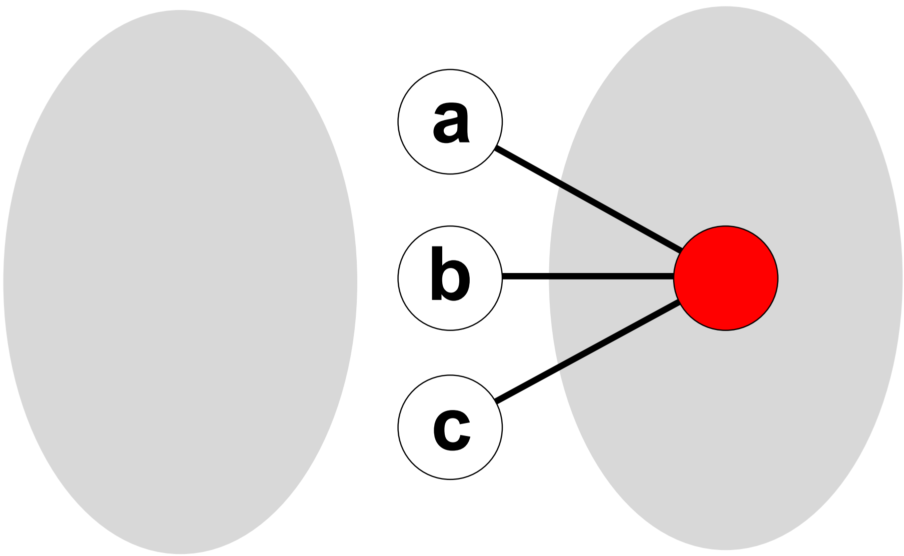
- Theorem [1963 Moon-Moser]:
  For infinitely many such graphs $G$, circ($G$) $\leq 9n^{\log_3 2}$ ($\log_3 2 \approx 0.631$).

- Theorem [2002 Chen-Yu]:
  For all such $G$, there is a positive constant $c$ such that circ($G$) $\geq cn^{\log_3 2}$.

- Theorem [1956 Tutte]:
  Every 4-connected planar graph $G$ is Hamiltonian.

# Between 3- and 4-connectivity

- A polyhedral graph G is essentially 4-connected if every 3-separator is the neighborhood of a single vertex.



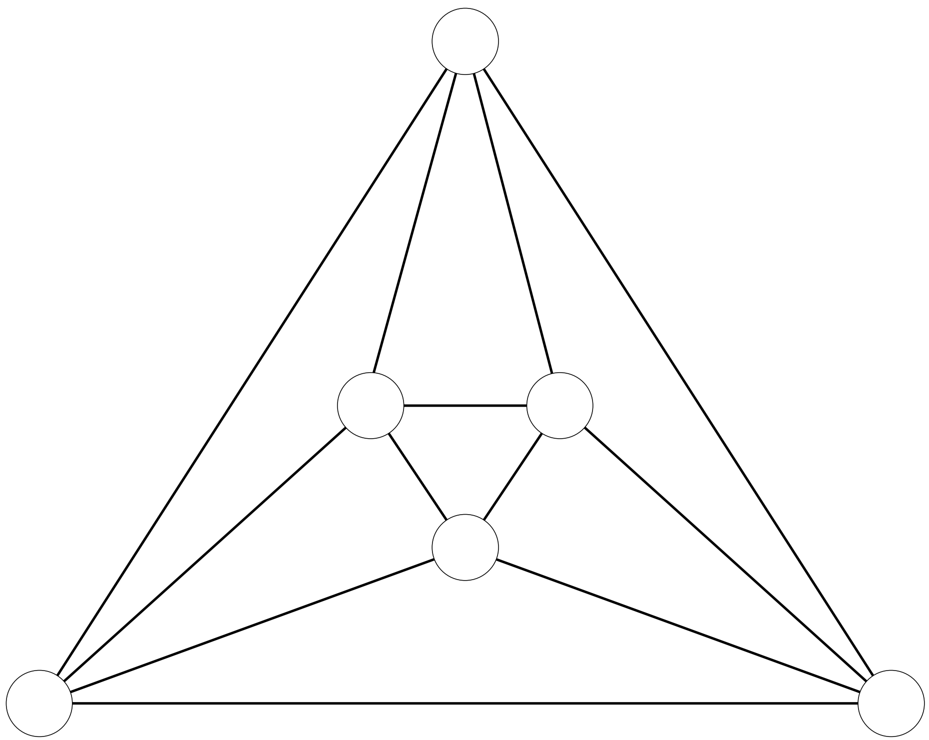

- Theorem [1992 Jackson-Wormald]:
  For every essentially 4-connected polyhedral graph G, $circ(G) \geq (2n+4)/5$.

- Theorem [1976 Grünbaum-Malkevitch]:
  For every cubic essentially 4-connected polyhedral graph G, $circ(G) \geq 3n/4$.

# But not Hamiltonian

Theorem [2016 Fabrici-Harant-Jendrol', 1992 Jackson-Wormald]:
For infinitely many (maximal planar) essentially 4-connected polyhedral graphs G, circ(G) ≤ 2(n+4)/3.

Proof sketch:



- Take any 4-connected maximal planar graph.

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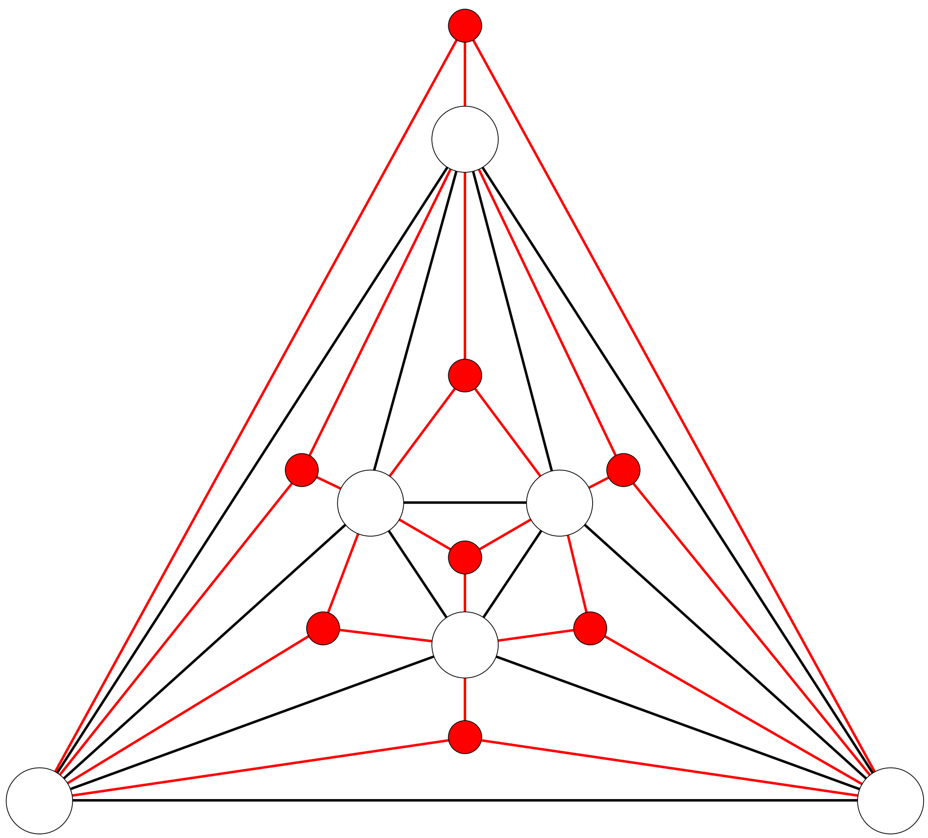

- Take any 4-connected maximal planar graph.
- Insert a vertex of degree 3 in every face; the graph is essentially 4-connected.

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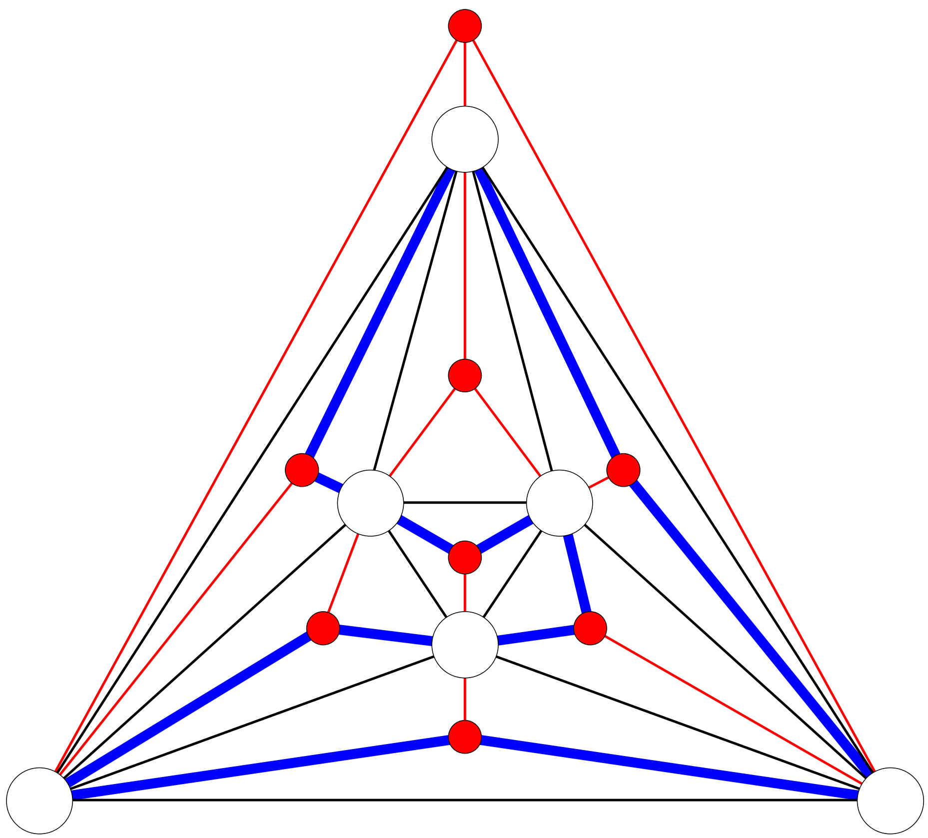
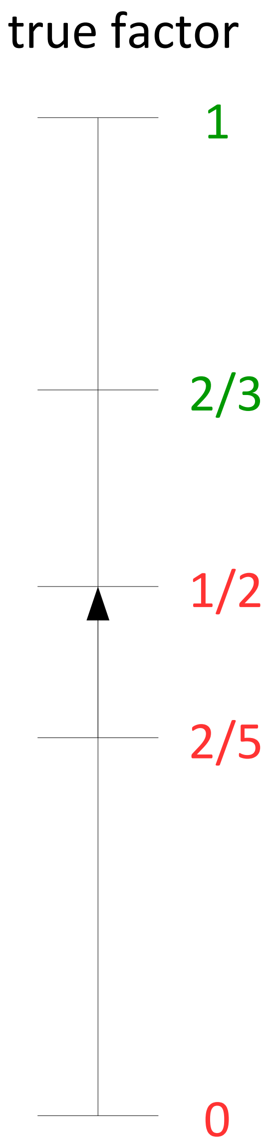


- As the red vertex set is independent, any cycle has at most twice the number of white vertices.
- As there are 2 = (n+4)/3-4 more red than white vertices, we miss 2 vertices.

# G polyhedral & essentially 4-connected

Theorem [2016 Fabrici-Harant-Jendrol']:
circ(G) $\geq (n+4)/2$

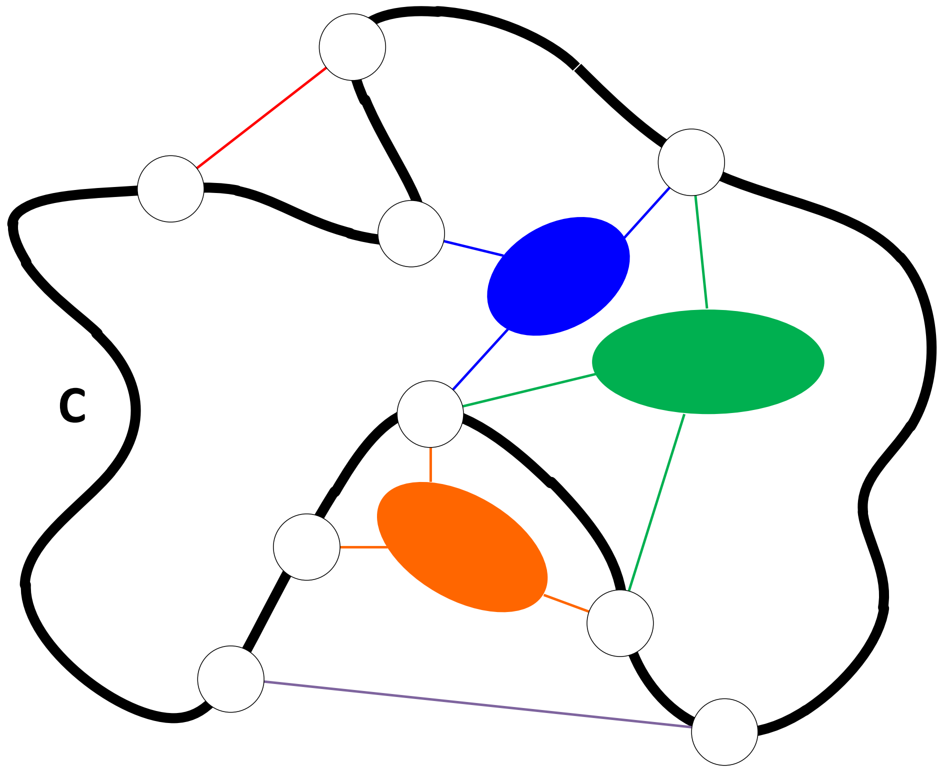


# Alternative proof using 2-walks

A cycle **C** of a plane 2-connected graph G is a **Tutte cycle** if

- every **C**-bridge has at most 3 attachments and
- every **C**-bridge containing an edge of the outer face has exactly 2 attachments.

An SDR of **C** is an SDR of the sets of attachment vertices of the **C**-bridges with 3 attachments.



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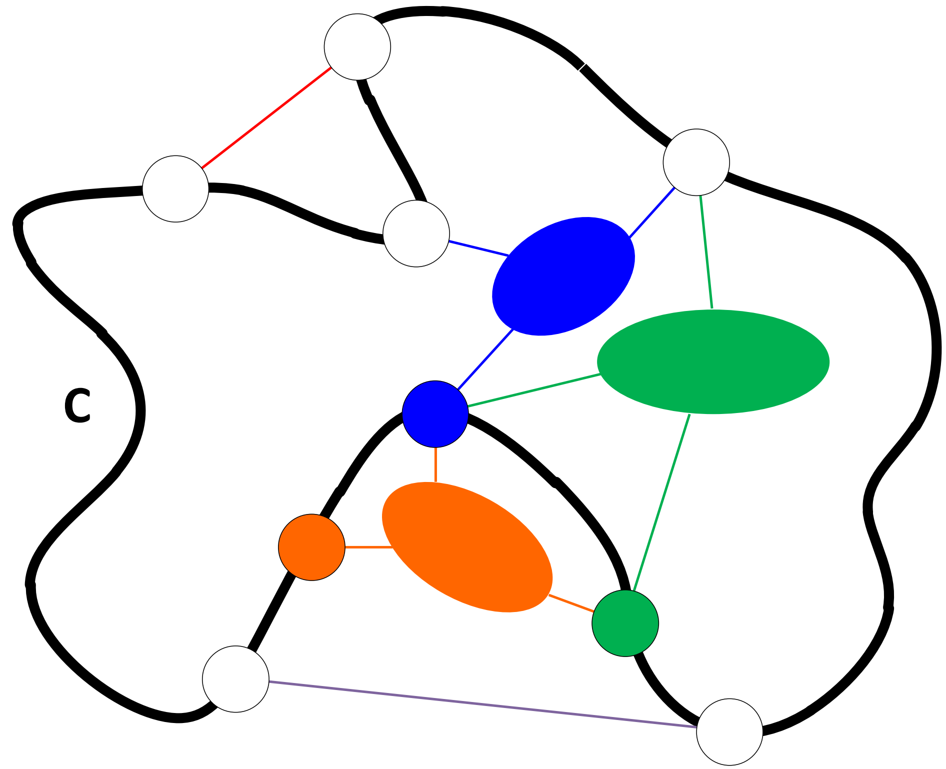
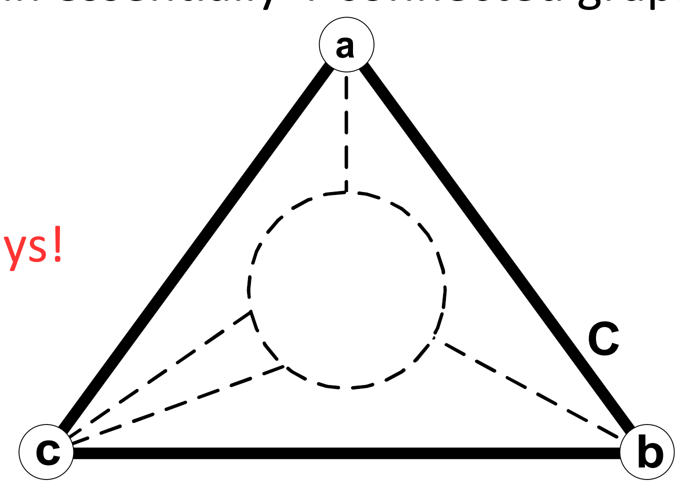
Theorem [1995 Gao-Richter-Yu]:
Every 3-connected plane graph has a 2-walk
(+ a Tutte cycle **C** for which an SDR exists).




Is **C** long in essentially 4-connected graphs?

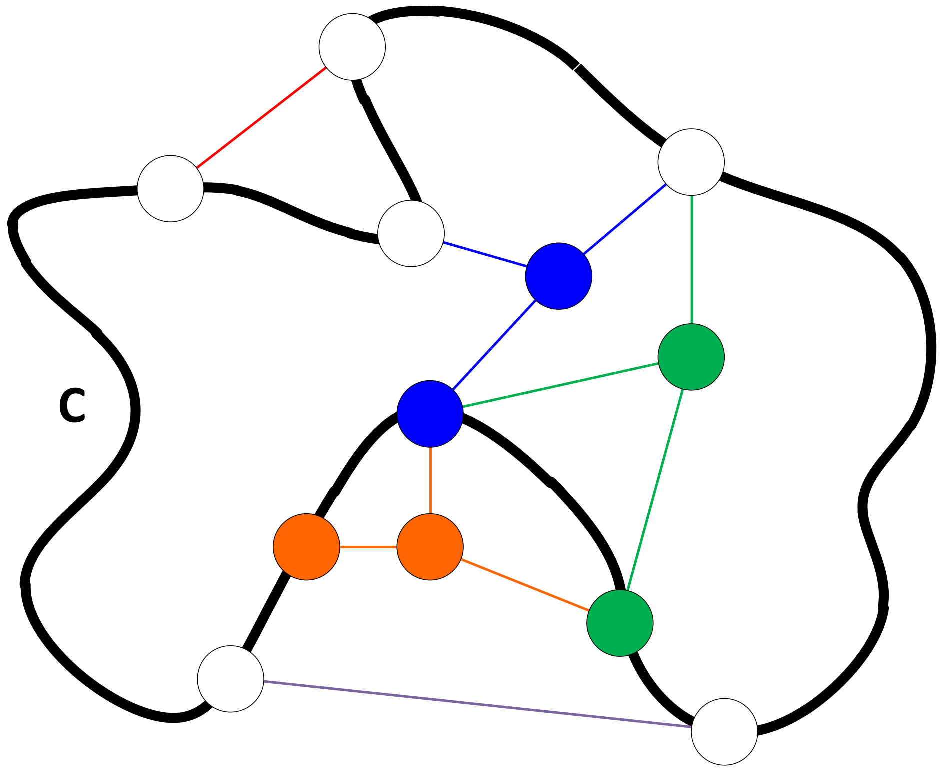
Not always!



# Alternative proof using 2-walks

But there is a Tutte cycle C for which an SDR of C exists such that V(G)-V(C) is independent!
(Trick: Choose an appropriate Tutte cycle of length at least 4.)

→ Every vertex not in C has degree 3.
→ $|V(C)| \geq |V(G)-V(C)|$
→ $|V(C)| \geq n/2$

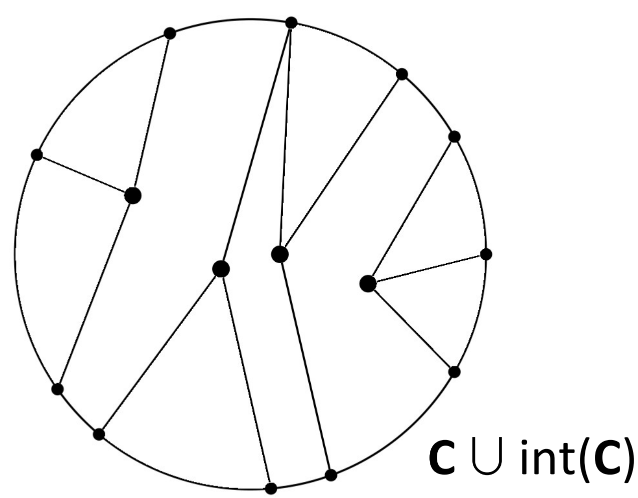


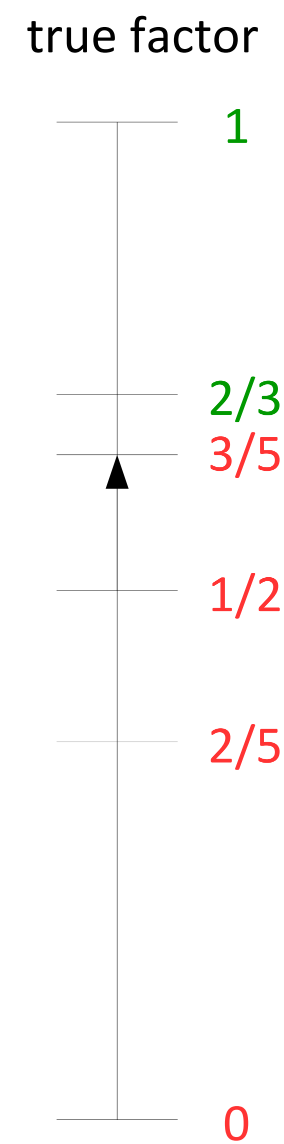
# New Result

Theorem:
For every essentially 4-connected polyhedral graph, circ(G) ≥ 3(n+2)/5.

Idea:

- Start with a longest Tutte cycle **C** such that
  - G-V(**C**) is independent and
  - no two consecutive vertices of **C** have a common neighbor outside **C**.
- Delete all chords of **C**.
- A face is green if it is incident to at most 1 vertex of G-V(**C**).



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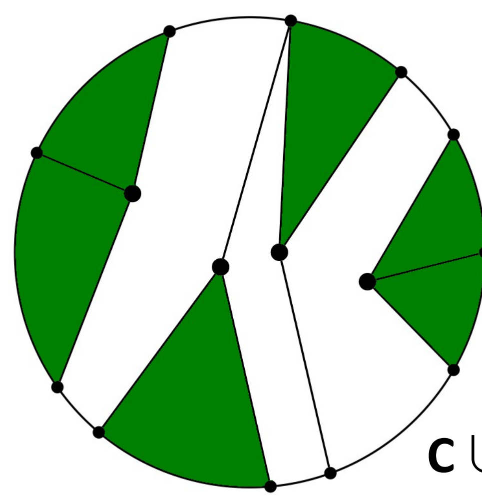
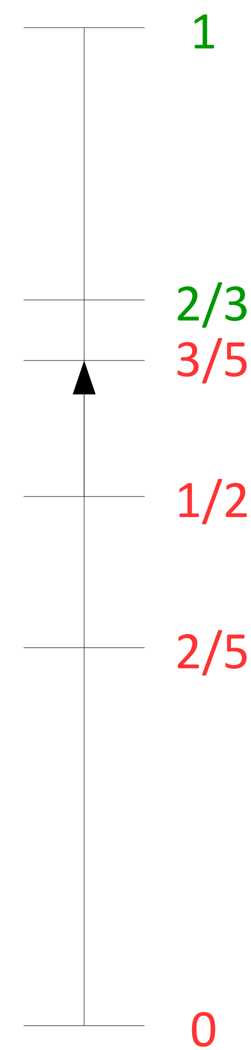
- Delete all chords of **C**.
- A face is green if it is incident to at most 1 vertex of G-V(**C**).
- The number $g$ of green faces satisfies $n+2-|V(\mathbf{C})| \leq g$.
- By discharging:
  - $g \leq 2|V(\mathbf{C})|/3$

→ claim



**C** ∪ int(**C**)

true factor



# Complexity

Theorem: For every essentially 4-connected polyhedral graph, a cycle of length at least 3(n+2)/5 can be computed in time $O(n^2)$.

Crucial parts of the previous proof:

instead, take a non-extendable one

- Start with a longest Tutte cycle **C** such that
  - G-V(**C**) is independent and
  - no two consecutive vertices of **C** have a common neighbor outside **C**.

Theorem [2017 Schmid, S.]: A Tutte path of a 2-connected graph (in the general Sanders-variant) can be computed in time $O(n^2)$.

Thank you!