Data Reduction and Combinatorics of the 3-Colorability Problem

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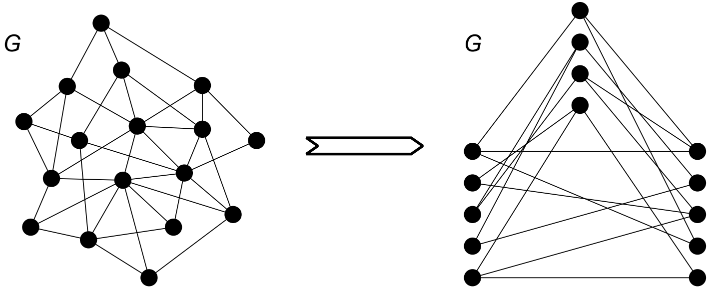
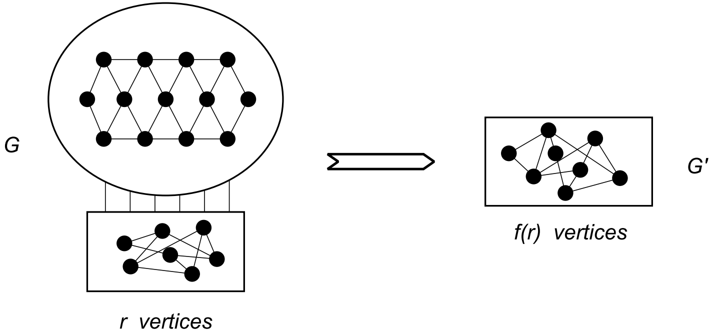
k-Colorability



Kernelization



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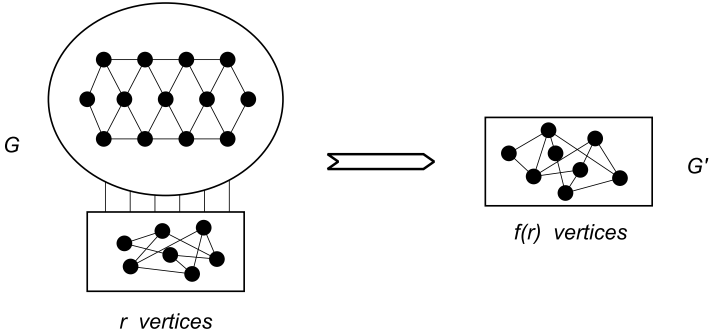

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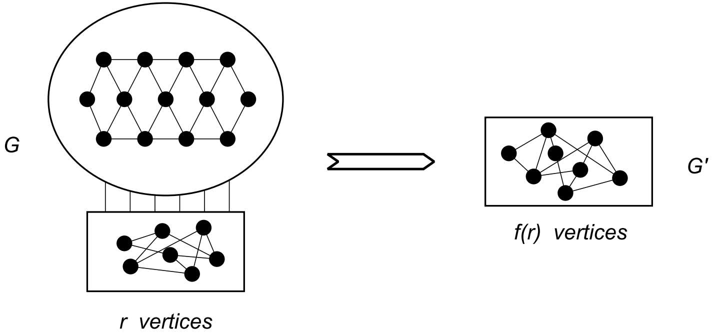

- assume that G can be made *nice* by removing r vertices
- a **kernelization** is a polynomial time algorithm that computes an equivalent instance G' of order $f(r)$

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Every graph class in which we can solve the **list k-colorability problem** in polynomial time: every vertex is equipped with a list $L(v) \subseteq \{1, \ldots, k\}$ of admissible colors.

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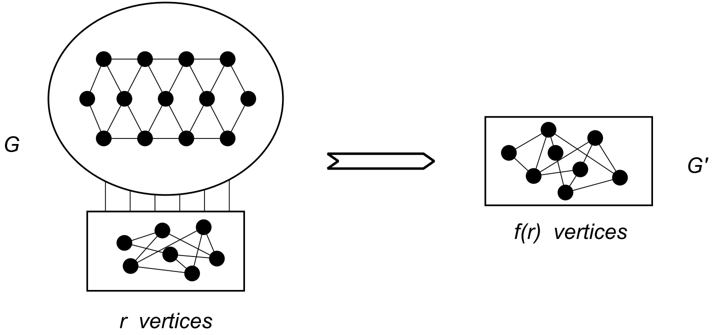
Theorem (Bonomo, Chudnovsky, Maceli, S, Stein, Zhong '14)
The list 3-colorability problem can be solved in polynomial time for P_7-free graphs.

Polynomial kernelization



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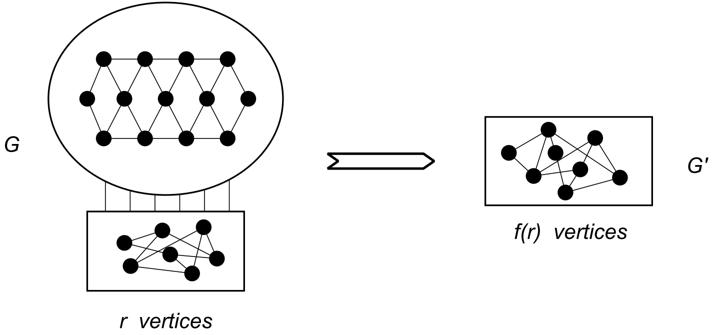
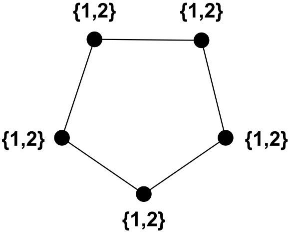


Problem

When is there a **polynomial kernelization**, meaning $f(r)$ is of order $r^{O(1)}$?

Polynomial kernelization

- let G be a graph and let $L(v) \subseteq \{1, \ldots, k\}$ for every vertex v



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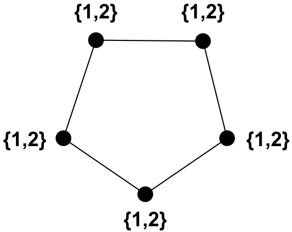

- we call (G, L) a **k-obstruction** if
  (a) there is no coloring for (G, L), and
  (b) there is a coloring if we remove any vertex from G

Polynomial kernelization

Theorem (Jansen & Kratsch 2013)

Let $\mathcal{P}$ be a graph property closed under deleting vertices. Assume there are only finitely many k-obstructions with property $\mathcal{P}$.

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If a property applies to all paths, there is no such polynomial kernelization.

Polynomial kernelization

Theorem (Chudnovsky, Goedgebeur, S, Zhong 2016)

Let H be a connected graph. The following two assertions are equivalent.

- *The 3-colorability problem admits a polynomial kernel of order $r^{O(1)}$ on the graphs that are H-free after removing r vertices.*
- *H is a path on at most six vertices.*

We assume that $NP \not\subseteq coNP/poly$.

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Open Problem

Characterize the existence of a polynomial kernel for arbitrary graph properties closed under vertex deletion.

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- gives a kernel of order $r^{3\cdot 2^{2^{104}}}$
- there are 1 441 407 obstructions on at most 9 vertices

Easy case: lists have size at most two

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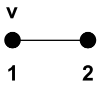

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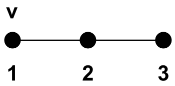

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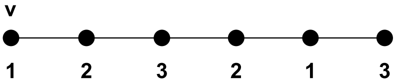

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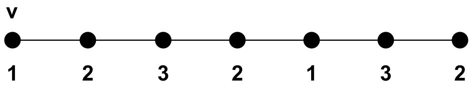

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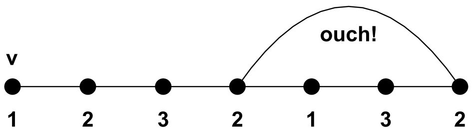

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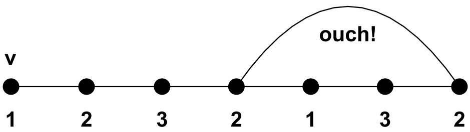

- doing this for both colors on v's list, we see that G is the union of two paths (not disjoint) starting in v

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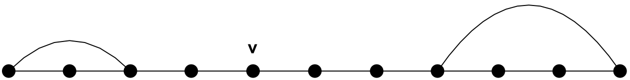

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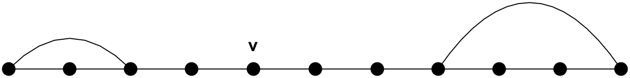

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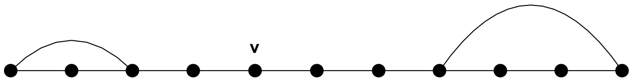

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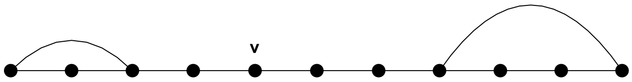

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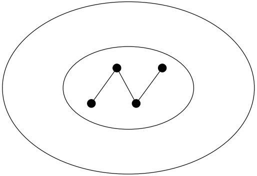

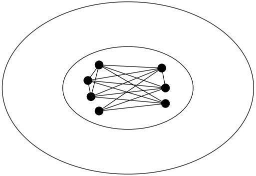

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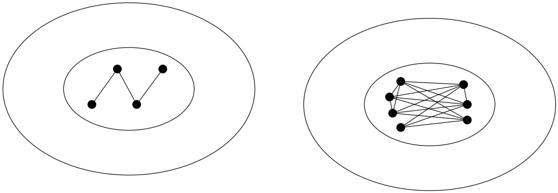

- in the first case we can win by guessing the coloring of the P_4

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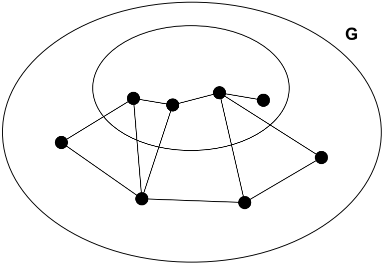

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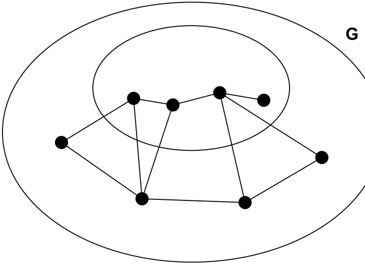

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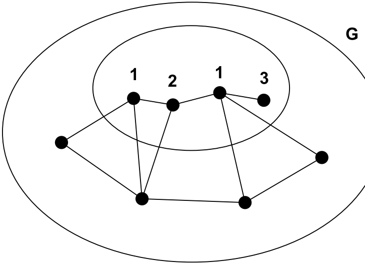

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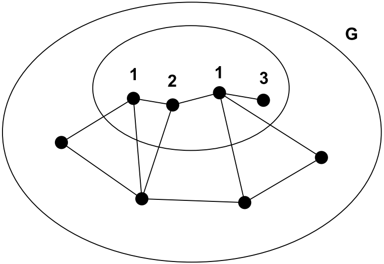

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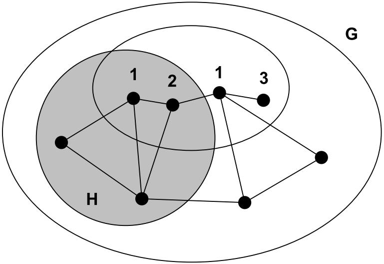

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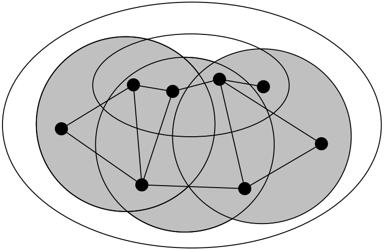

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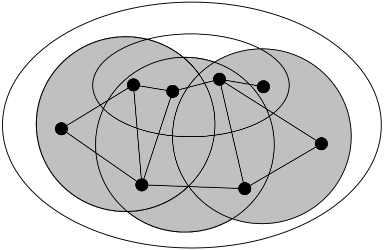

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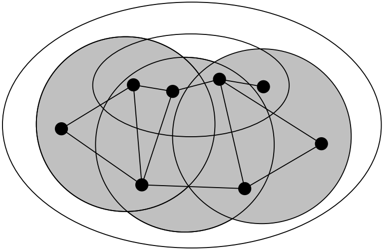

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- we have constantly many H's, so G has a bounded order

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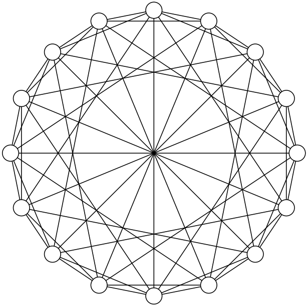

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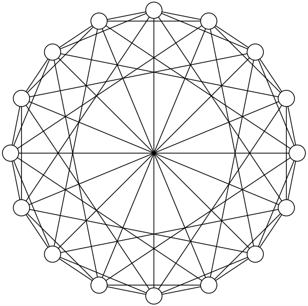

- Easy: infinte familes of claw-free obstructions, and obstructions of large girth
- If H is connected and not a subgraph of P_6, there are infinitely many obstructions in the class of H-free graphs