New Crossing Lemmas

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The crossing number $cr(G)$ of a graph G is the minimum number of edge crossings in any drawing of G in the plane.

$cr(G) = 0 \iff G$ is planar

$cr(G) \leq 1 \iff G$ does not have $H_1, \ldots, H_{41}$ as a minor

(Robertson-Seymour 1993)

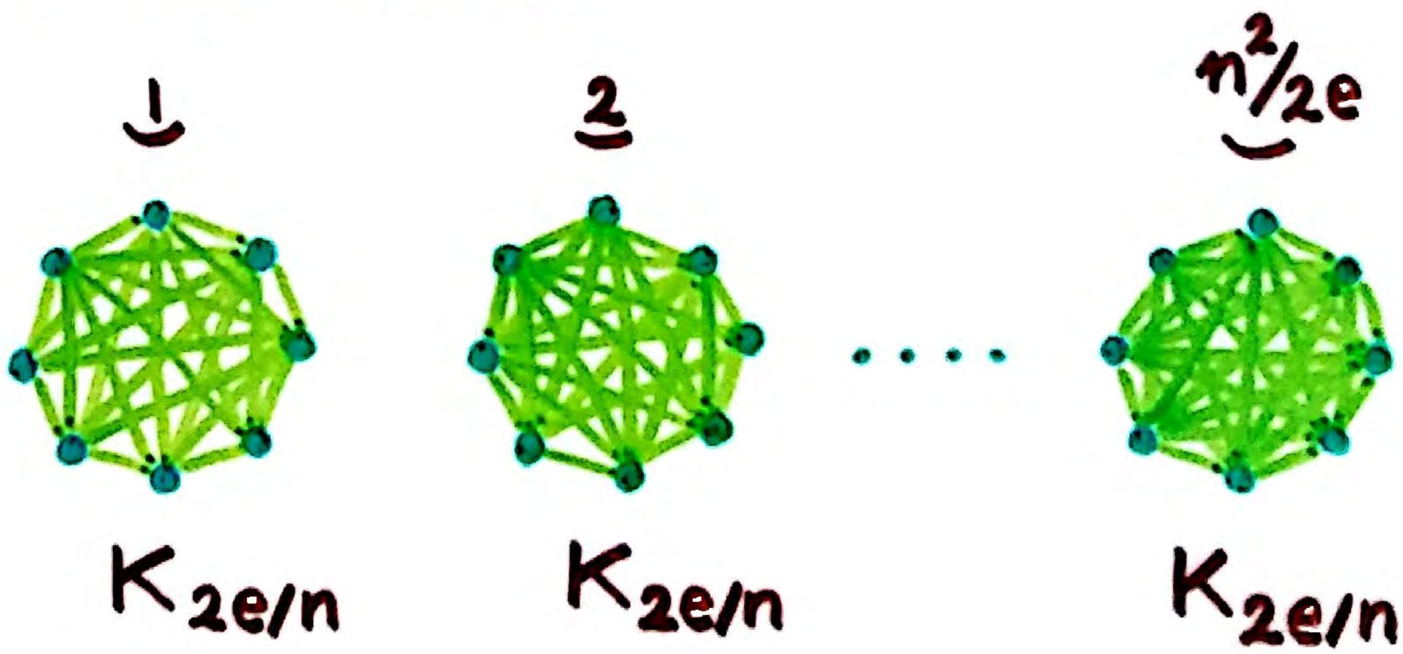
$cr(G) \leq 2 \iff$??

Theorem (Ajtai-Chvátal-Newborn-Szemerédi 1982, Leighton 1983)

For every simple graph with n vertices and $e \geq 4n$ edges,

$$cr(G) \geq \frac{1}{64} \frac{e^3}{n^2}$$

Construction



Lemma

$$cr(G) \geq e-(3n-6) > e-3n$$

Proof of Theorem

Pick each $v \in V(G)$ with probability p, and let $G' \subseteq G$ denote the subgraph induced by these points.

$$E[cr(G')] > E[e'] - 3E[n']$$

$E[e']$: #edges of G'

$E[n']$: #vertices of G'

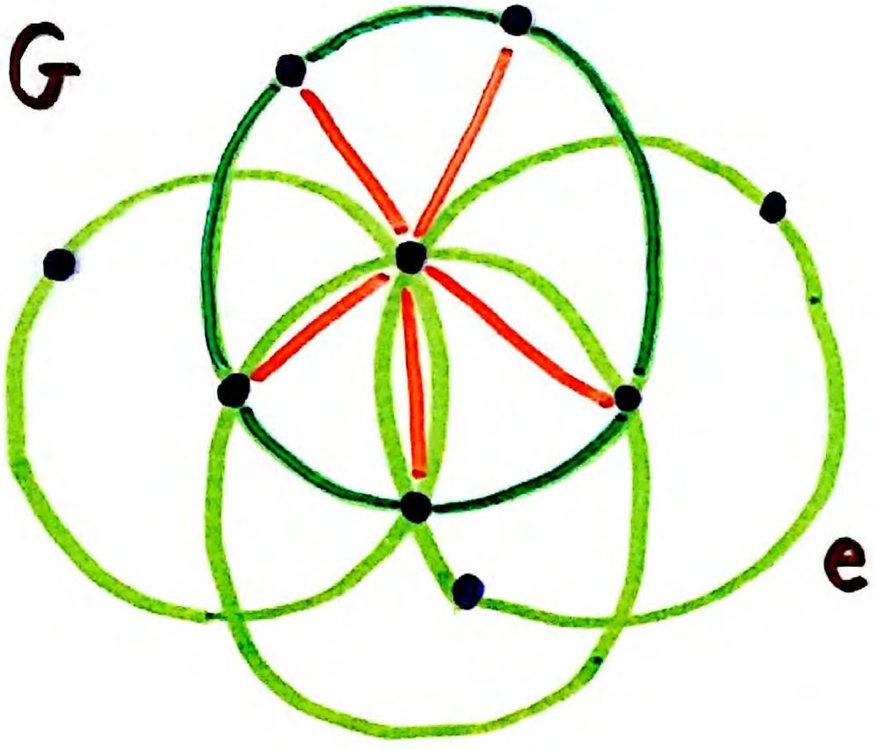
$$p^4 cr(G) > p^2 e - 3pn$$

Set $p = 4n/e$.

APPLICATIONS

I. **Theorem** (Spencer-Szemerédi-Trotter)

The number of unit distances determined by n points in the plane is $O(n^{4/3})$.



u unit distances

$2u$ incidences between circles & points

$e = 2u$ edges (?!)

n vertices

$$2\binom{n}{2} \geq cr(G) \geq \frac{1}{33.75}\,\frac{u^3}{n^2}$$

APPLICATIONS

II. Given a set of n points in the plane, a segment connecting 2 of them is a halving edge if the number of points on its left side is the same as on its right side.
(n even, no 3 collinear points)

What is the maximum number of halving edges ?

$\leq cn^{3/2}$ Lovász 1971
Erdös-Lovász-Simmons -Straus 1973

$\leq c\frac{n^{3/2}}{\log^* n}$ P.-Steiger-Szemerédi 1992

APPLICATIONS

II. Given a set of n points in the plane, a segment connecting 2 of them is a halving edge if the number of points on its left side is the same as on its right side.
(n even, no 3 collinear points)

Theorem (Dey 1998)
The maximum number of halving edges of an n-element point set in the plane is $\leq c n^{4/3}$.

MONOTONE PROPERTIES

G satisfies P, $H \subseteq G \Longrightarrow$ H satisfies P

G_1, G_2 satisfy $P \Longrightarrow$ $G_1 \dot{\cup} G_2$ satisfies P

Theorem (P.-Spencer-Tóth 2000)
Let G be a graph with n vertices and $e \geq 4n$ edges that does not contain a $K_{2,2}$. Then

$$cr(G) \geq c \frac{e^4}{n^3}$$

$$G \not\supseteq K_{r,r} \Longrightarrow cr(G) \geq c_r \frac{e^{3+1/r-1}}{n^{2+1/r-1}}$$

MULTIGRAPHS

Theorem (Székely 1997)

Let G be a multigraph with n vertices, $e \geq 4mn$ edges and maximum edge multiplicity m. Then

$$cr(G) \geq \frac{1}{64} \frac{e^3}{mn^2}$$

This bound cannot be improved.

Proof

Keep at most one edge between any pair of vertices, each with probability $\frac{1}{m}$.

SIMPLE TOPOLOGICAL MULTIGRAPH

G has a drawing in the plane such that graph

- any two independent edges cross $\leq$ once
- no two edges that share 1 or 2 endpoints cross
- there is ≥ 1 vertex inside and outside every region enclosed by two parallel edges

Lemma. Let G be a planar simple multigraph with n vertices and e edges. Then

$$e \leq 3n - 6$$

$cr_s(G)$ crossing number of the simple topological multigraph G

$cr_s(G) = 2 \geqslant cr(\bar{G}) = 0$

Theorem (P. - Tóth 2017)

Let G be a simple topological multigraph with n vertices and $e \geqslant 4n$ edges. Then

$$cr_s(G) \geqslant \frac{1}{10^3} \frac{e^3}{n^2}$$

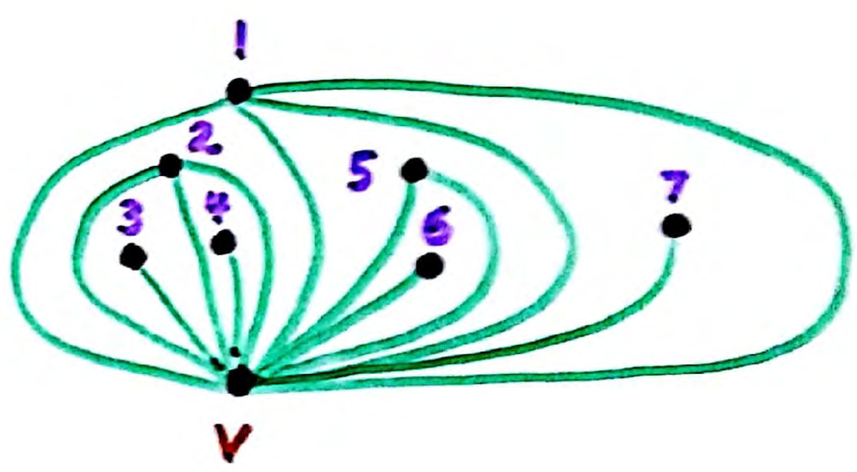
This bound cannot be improved.

Lemma. Let G be a simple topological multigraph with n vertices and e edges. Then

$$e < n^2$$

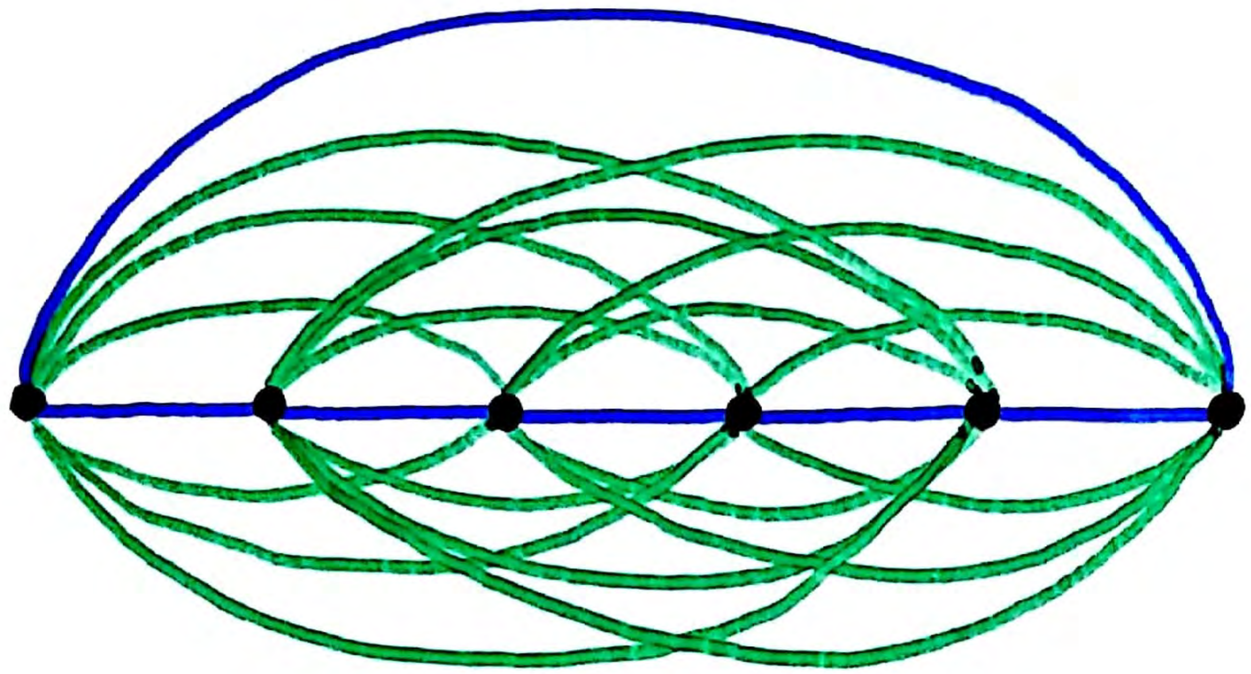
Proof.

$d(v) \leq 2n-3 \quad (\forall v \in V(G))$



No ... a ... b ... a ... b ...

1 2 3 2 4 2 1 5 6 5 1 7 1 ...



$$e = 2\binom{n}{2} - n = n(n-2)$$

Crossing Number and Bisection Width

$b(G)$ bisection width : minimum number of edges whose removal cuts G into two subgraphs, each having $\geq \frac{1}{3}|V(G)|$ vertices.

Theorem (P.-Shahrokhi-Szegedy 1996)
Let G be a graph with n vertices of degree $d_1, \dots, d_n$. Then

$$b(G) \leq 10\sqrt{cr(G)} + 2\sqrt{\sum_{i=1}^{n} d_i^2}$$

$\Longrightarrow$ Crossing Lemma

Crossing Number and Bisection Width for Simple Topological Multigraphs

$b_s(G)$ bisection width of the simple topological multigraph G : minimum number of edges whose removal splits G into disjoint simple topological multigraphs G_1, G_2 with $|V(G_1)|, |V(G_2)| \geq \frac{1}{3}|V(G)|$

Theorem (P.-Tóth 2017)
Let G be a simple topological multigraph with n vertices of degree $d_1, \dots, d_n$. Then

$$b_s(G) \leq 10^3 \sqrt{cr_s(G)} + 10^3 \sqrt{\sum_{i=1}^{n} d_i^2}$$

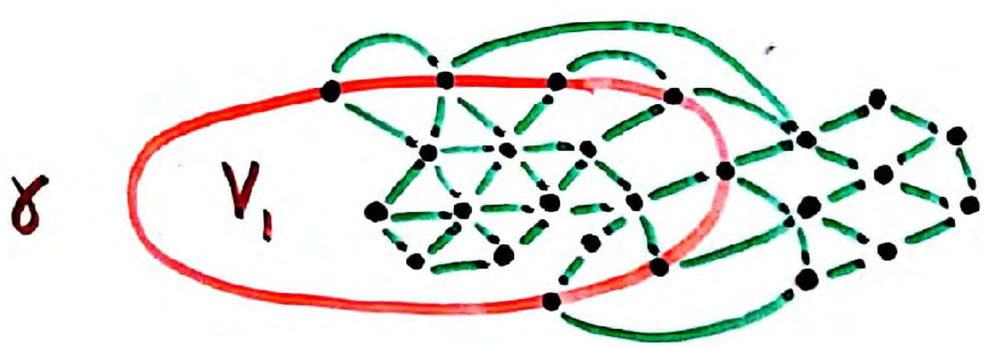
Separator Theorem for Planar Graphs

Theorem (Alon-Seymour-Thomas 1994)

Given any plane graph G with weight function $w : V(G) \to \mathbb{R}_{\geq 0}$, there is a closed curve γ that does not meet the interior of any edge such that

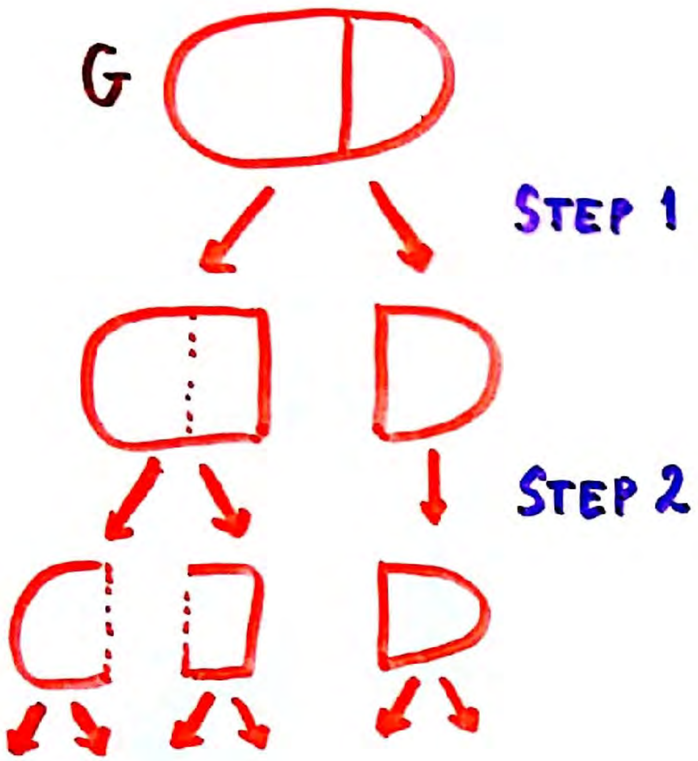
- $|\gamma \cap V(G)| \leq 3\sqrt{|V(G)|}$
- if V_1 and V_2 denote the subsets of vertices in the interiors of the two regions bounded by γ, then

$$w(V_i) + \frac{1}{2} w(\gamma \cap V(G)) \leq \frac{2}{3} w(V(G)) \qquad i = 1, 2$$



$$w(V_i) = \sum_{v \in V_i} w(v)$$

Decomposition Algorithm



- After Step i, each piece has $\leq \left(\frac{2}{3}\right)^i n$ vertices

  if $\left(\frac{2}{3}\right)^i n < \frac{1}{2}\frac{e}{n}$, then STOP

  otherwise SPLIT pieces of size $> \left(\frac{2}{3}\right)^{i+1} n$

- If $cr_s(G) \leq \frac{1}{10^3}\frac{e^3}{n^2}$, the total number of deleted edges is $\leq \frac{e}{2}$

- # remaining edges

$$\sum_{j=1}^{m} e(G_j) < \sum_{j=1}^{m} n^2(G_j) < \sum_{j=1}^{m} n(G_j)\frac{1}{2}\frac{e}{n}$$

$$< \frac{e}{2}$$

Contradiction.