Extension of graphs on surfaces to 3-colorable triangulations

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Extension of graphs on surfaces

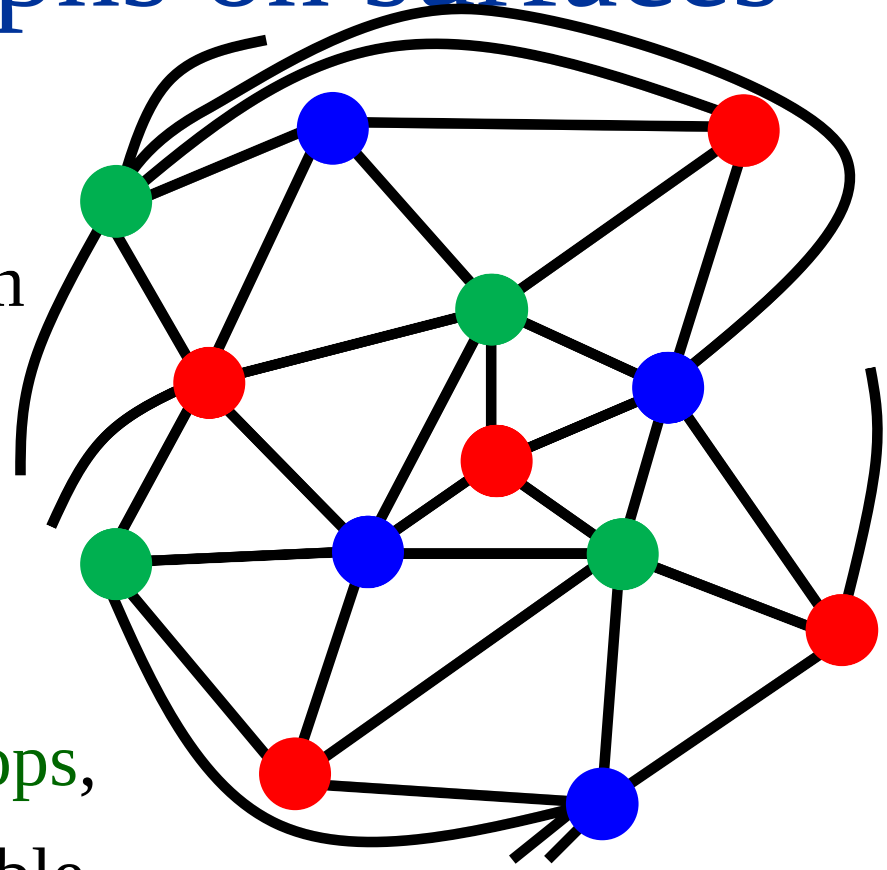
G : graph on a surface

Extension of G to triangulation

$\Leftrightarrow$

Obtain a tri. by adding edges

(c.f. maximal plane graph)



Allowing multiple edges and loops,

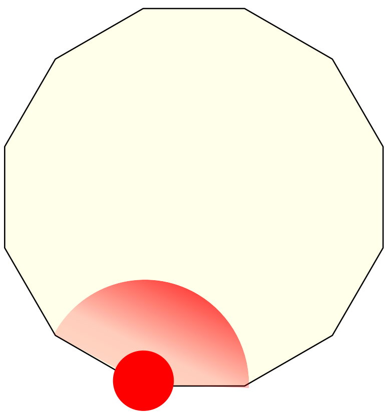
this is always possible.

Find an extension to a 3-colorable triangulation

Art Gallery Problem

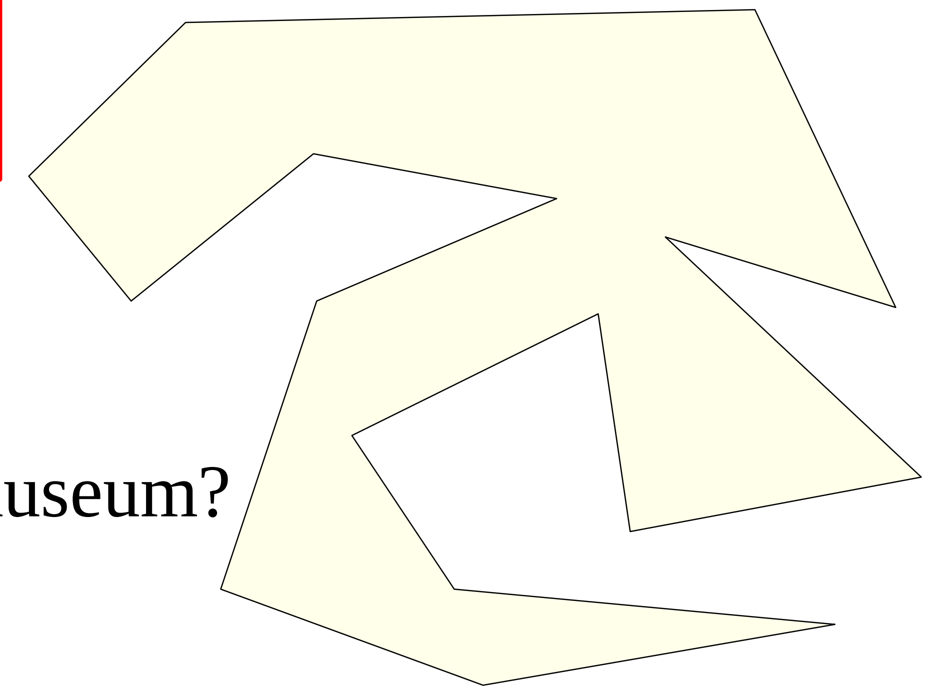
Art Gallery Problem :

How many guards do we need
to watch everywhere in a museum?



What about

a non-convex museum?



One guard is enough

Art Gallery Problem

<u>Art Gallery Problem :</u>

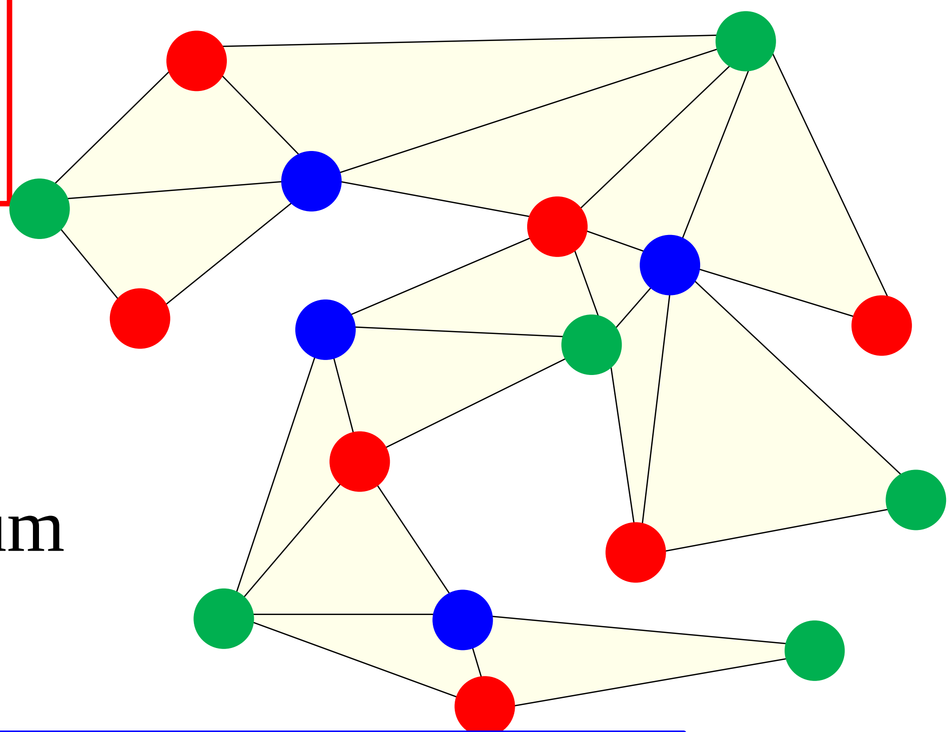
How many guards do we need
to watch everywhere in a museum?

(Chvatal, \`75)

$n/3$ guards is enough

n : # of walls in the museum

<u>Short proof by Fisk in \`78</u>



Key idea:

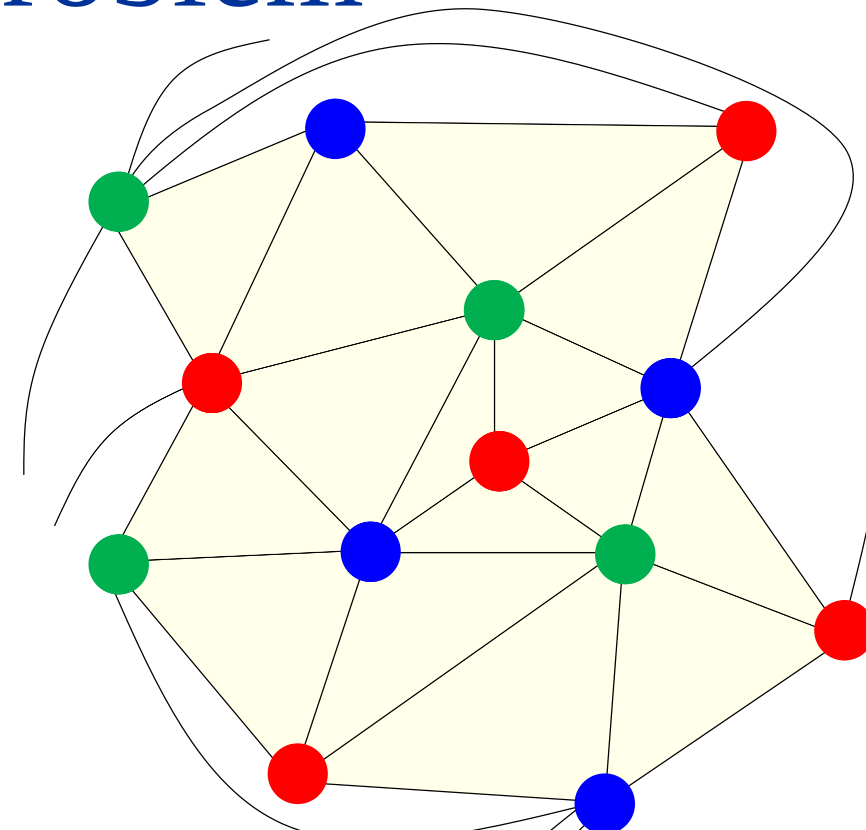
Extension to a 3-colorable disk triangulation

Prison Problem

<u>Prison Problem :</u>

How many guards do we need
to watch everywhere in a <u>prison</u>?

It has already rooms



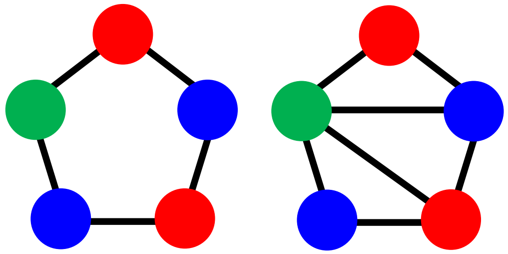
To obtain $n/3$ bound,
consider an extension to 3-colorable triangulation

Extension Problem

Extension Problem :

Determine if a graph can be extended to a 3-colorable tri.

General plane graphs



NP-complete even for pentagulations

(Ext. prob. for pentagulations = 3-coloring prob.)

Quadrangulations (all faces are quadlangular)

$\forall$plane quad. has an extension to a 3-colorable triangulation.

(Hoffmann & Kriegel, `96, He & Zhang, `05)

Extension Problem

Improve this in the following two senses.

- ✓ From the plane to other surfaces

  (projective plane, torus, Klein-bottle, etc.)

- ✓ Mosaics (all faces are tri- or quadrangular)

Goal: A necessary and sufficient condition

for mosaics on a surface

to have an extension to 3-colorable triangulation

Extension Problem

Mosaics (all faces are tri- or quadrangular)

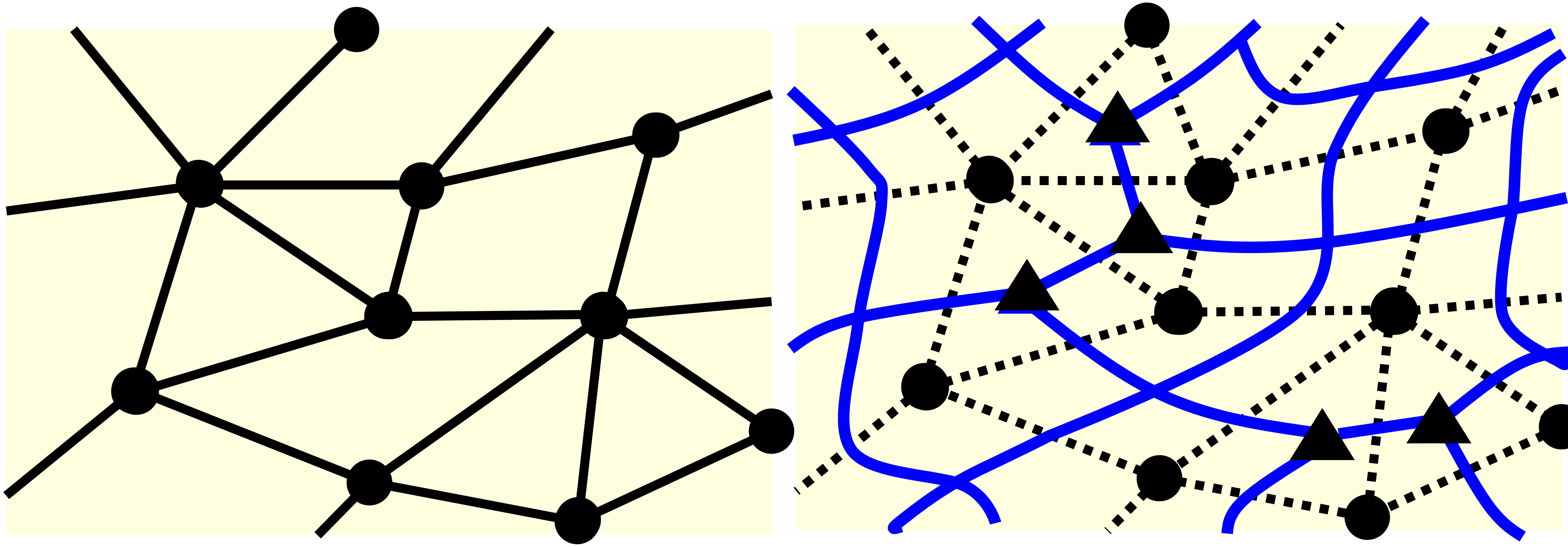
$$\begin{array}{ccc} \text{A mosaic } G & \longrightarrow & \text{3-colorable tri.} \\ \downarrow & & \uparrow \\ \text{Eulerian tri.} & \longrightarrow & \boxed{\begin{array}{c}\text{Monodoromy} \\ \mathbb{Z}_2 \text{ part} \longrightarrow \mathbb{Z}_3 \text{ part}\end{array}} \end{array}$$

<u>Fact:</u>

T : tri. of a surface   3-colorable $\Rightarrow$ T : Eulerian

The converse holds only when the planar case

S-dual and $\mathbb{Z}_3$-orientation



S-walk of G^*: a walk passing thr. vertices of deg. 4 straightly

(It might cross other S-walks, or itself.)

S-dual and $\mathbb{Z}_3$-orientation

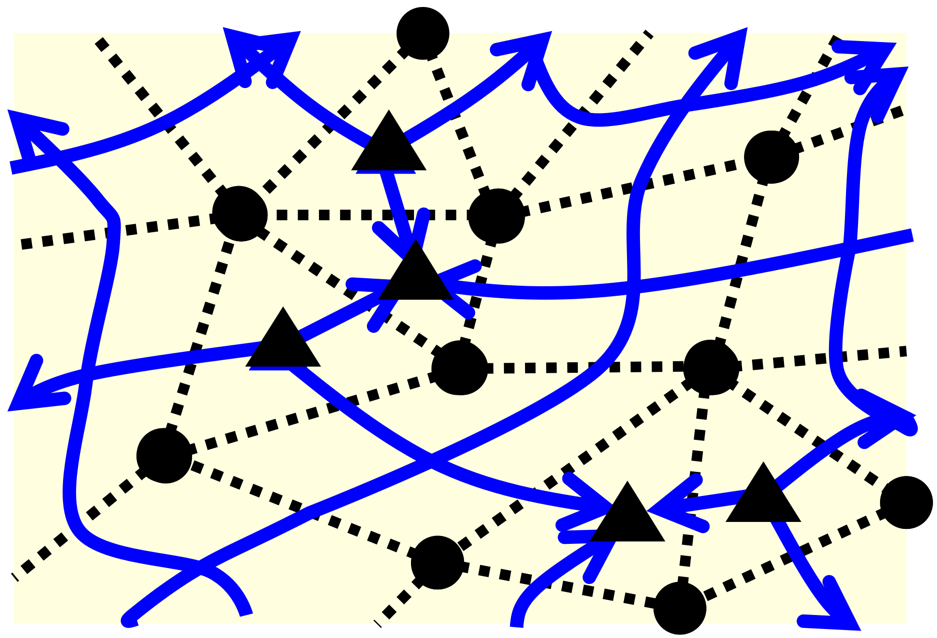
$S(G^*)$: the S-dual of mosaic G

$V(S(G^*))$: the set of tri. of G

$E(S(G^*))$: the set of S-walks

2 tri.s are adjacent

$\Leftrightarrow$ they are conn. by an S-walk



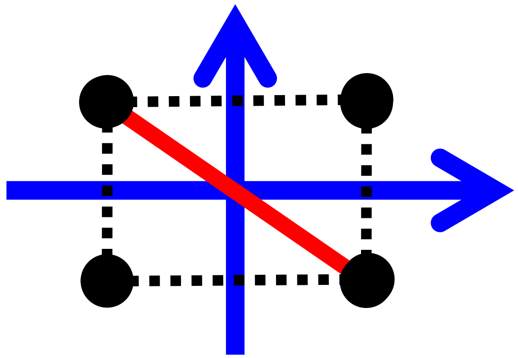
Key: $\mathbb{Z}_3$-orientation of S-dual produces an Eulerian tri.

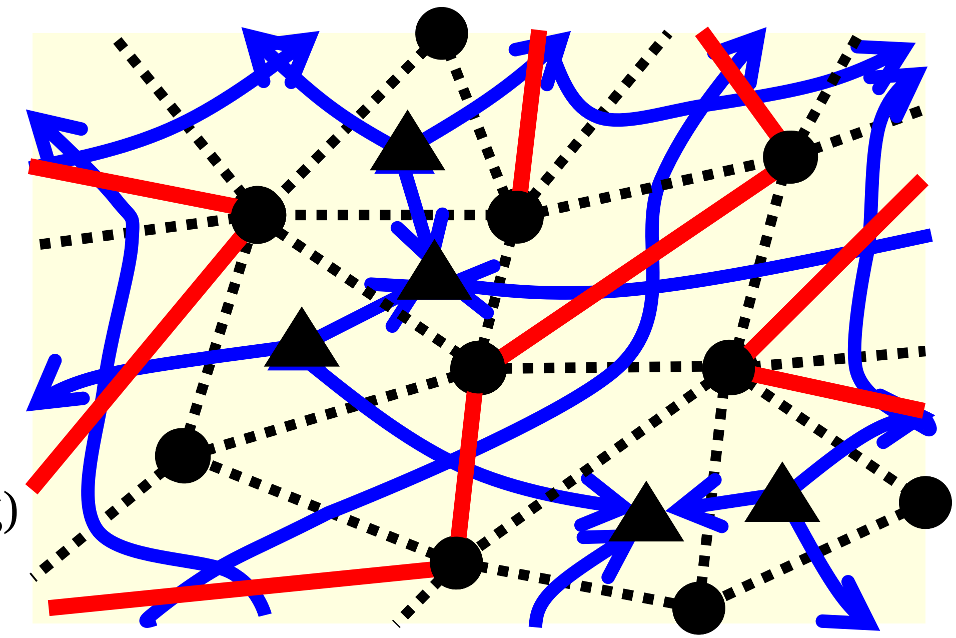
ori. with (in-deg.) $-$ (out-deg.) $\equiv 0 \pmod 3$

S-dual and $\mathbb{Z}_3$-orientation

Add diagonal to each quad.

s.t.





This gives an Eulerian tri. (He & Zhang)

Why? (homework)

Key: $\mathbb{Z}_3$-orientation of S-dual produces an Eulerian tri.

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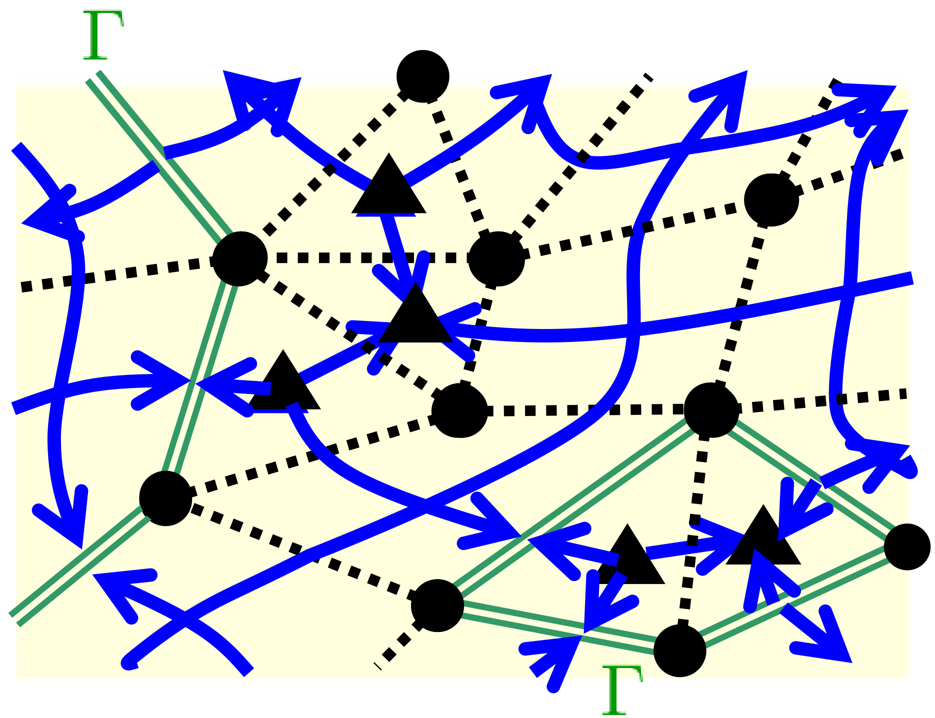
S-dual and $\mathbb{Z}_3$-orientation

Γ : cycles of the primal graph G



Γ-subdivided $\mathbb{Z}_3$-orientation

$\Leftrightarrow$

$\mathbb{Z}_3$-ori. changing the direction at each point belonging to Γ

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S-dual and $\mathbb{Z}_3$-orientation

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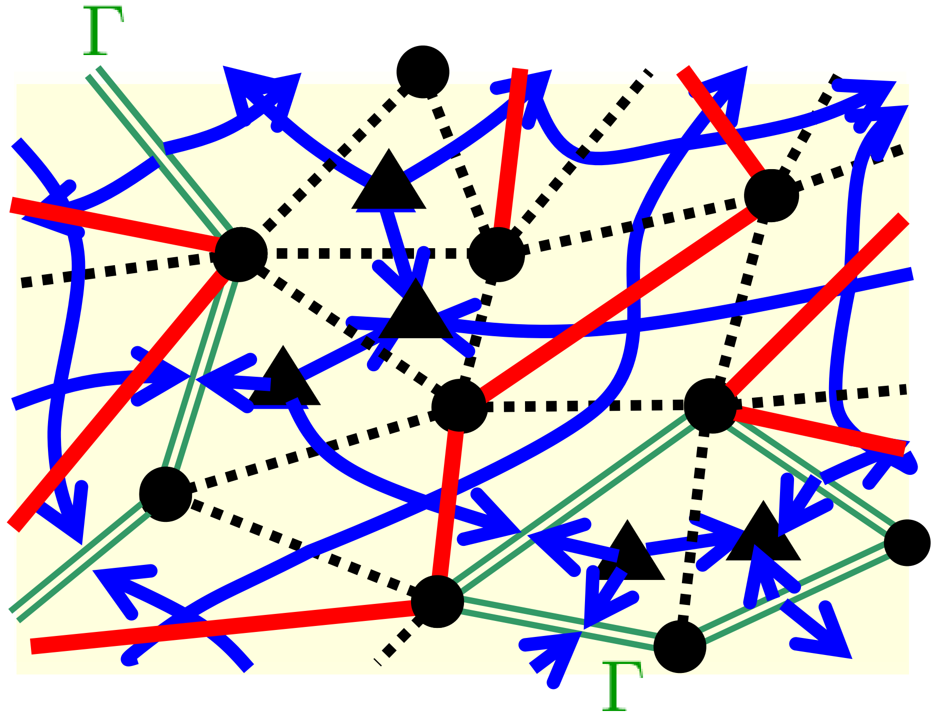

Γ-subdivided $\mathbb{Z}_3$-orientation

$\Leftrightarrow$

$\mathbb{Z}_3$-ori. changing the direction at each point belonging to Γ

- ✓ $\forall\,\Gamma$-sub. $\mathbb{Z}_3$-ori. also produces an Eulerian tri.
- ✓ $\forall$ extensions to Eulerian tri. are expressed as above for $\exists\,\Gamma$
- ✓ If Γ is not contractible, it produces another triangulation

(NNO, `15)

Extension Problem

Mosaics (all faces are tri- or quadrangular)

A mosaic G $\longrightarrow$ 3-colorable tri.

Eulerian tri. $\longrightarrow$

Monodoromy

$\mathbb{Z}_2$ part $\longrightarrow$ $\mathbb{Z}_3$ part

<u>Fact:</u>

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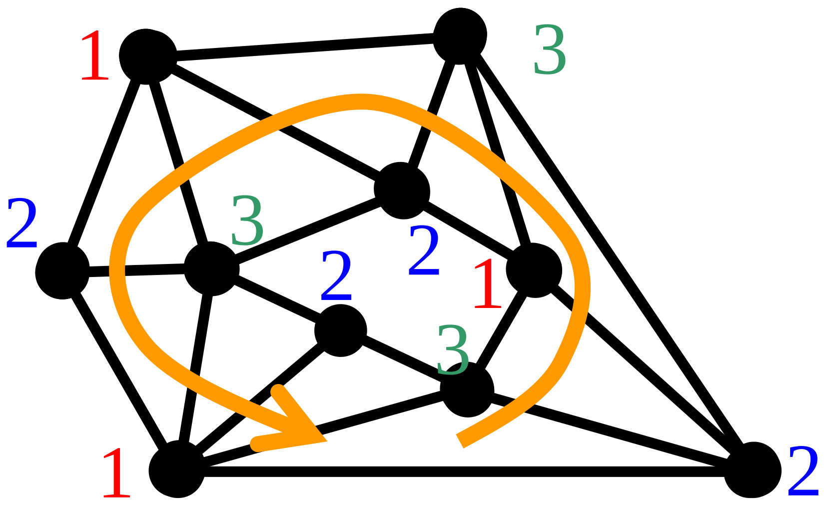
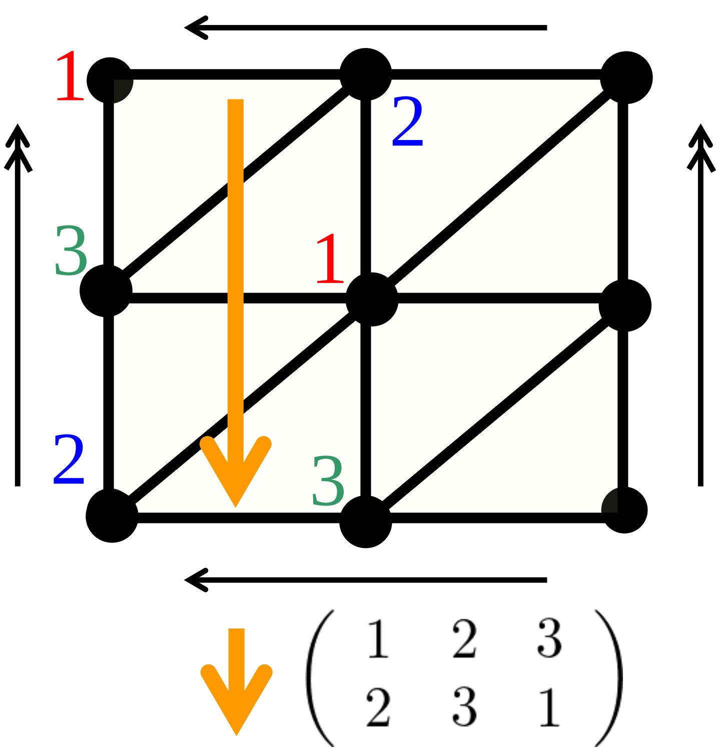
The converse holds only when the planar case

Monodromy

T: Eulerian triangulation of F^2



Identity map (id)



$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

<u>Claim</u>

For $\forall$ contractible curve, the gap along it is id.

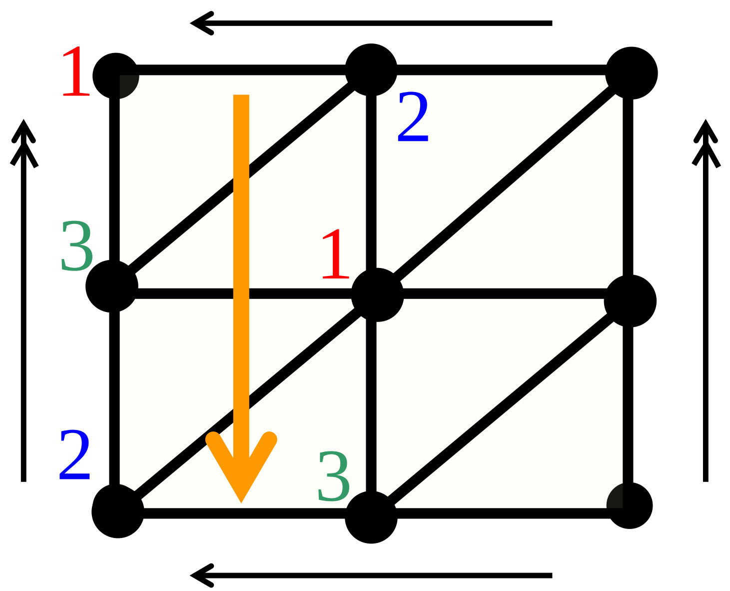
Monodromy

T : Eulerian triangulation of F^2

The fundamental group of F^2

$\sigma_T : \pi_1(F^2) \to S_3$: monodromy

The symmetric group of degree 3



$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

Claim

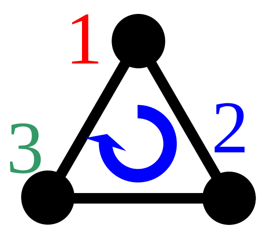
Eulerian tri. T is 3-colorable $\Leftrightarrow$ $\sigma_T(\gamma) = \mathrm{id}$ for $\gamma \in \pi_1(F^2)$

Monodromy

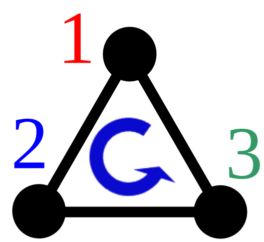
T : Eulerian triangulation of F^2

If T is 3-colorable,

$\forall$ curve alternatively passes thr.

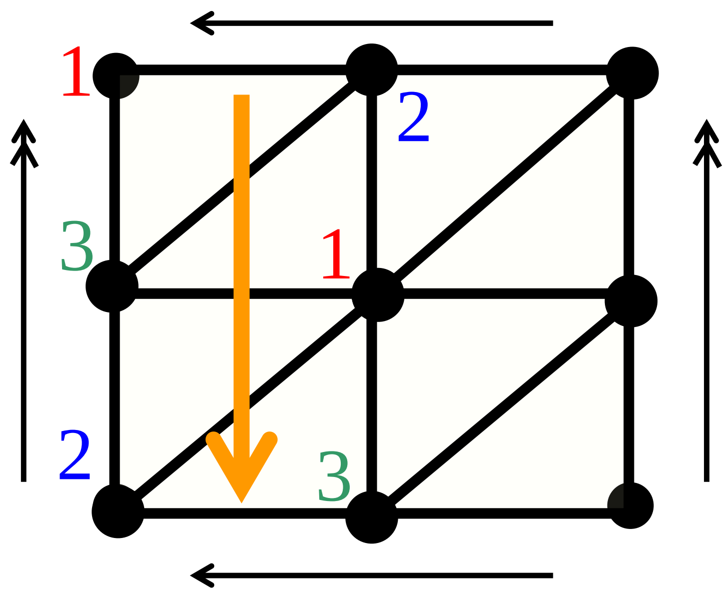


and



Face-2-coloring if F^2 : orientable

(call it an H-face-2-coloring)



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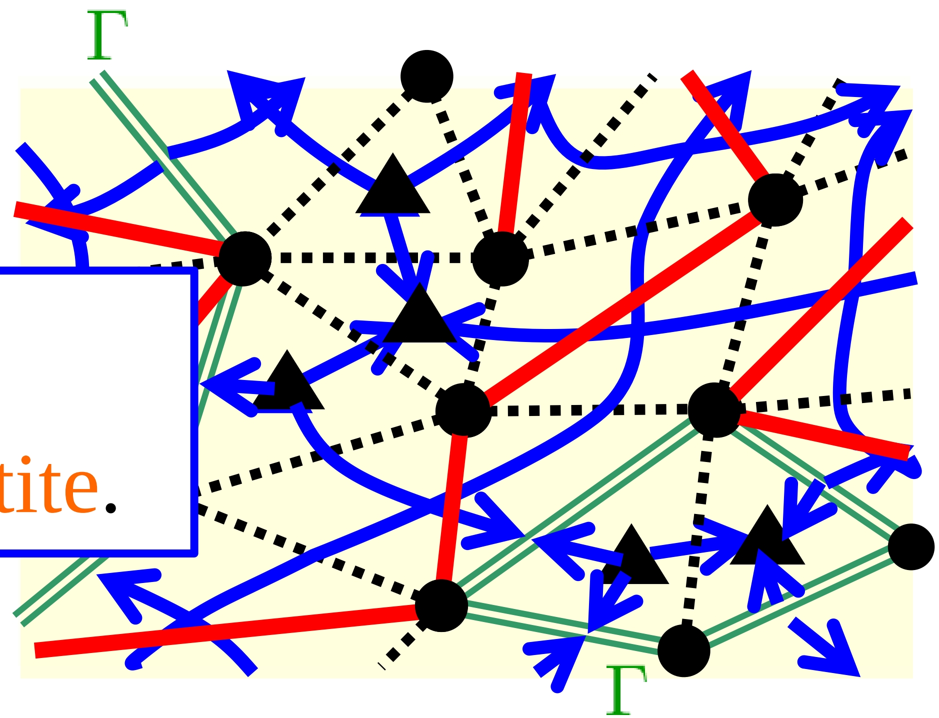
Eulerian tri. T is 3-colorable $\Leftrightarrow$ $\sigma_T(\gamma) = \mathrm{id}$ for $\gamma \in \pi_1(F^2)$

$\mathbb{Z}_2$ part of monodoromy

Γ : cycles of the primal graph G



Γ-subdivided $\mathbb{Z}_3$-orientation

$\Leftrightarrow$

$\mathbb{Z}_3$

at each point belonging to Γ

For orientable case,

S-dual has to be bipartite.

Find suitable Γ-sub. $\mathbb{Z}_3$-ori. guaranteeing H-face-2-colorable

Γ : $\mathbb{Z}_2$-homologous to
- an empty graph (orientable case)
- an orientizing (nonorientable case)

Extension Problem

Mosaics (all faces are tri- or quadrangular)

$$\begin{array}{ccc} \text{A mosaic } G & \longrightarrow & \text{3-colorable tri.} \\ \downarrow & & \uparrow \\ \text{Eulerian tri.} & \longrightarrow & \boxed{\begin{array}{c}\text{Monodoromy} \\ \mathbb{Z}_2 \text{ part} \longrightarrow \mathbb{Z}_3 \text{ part}\end{array}} \end{array}$$

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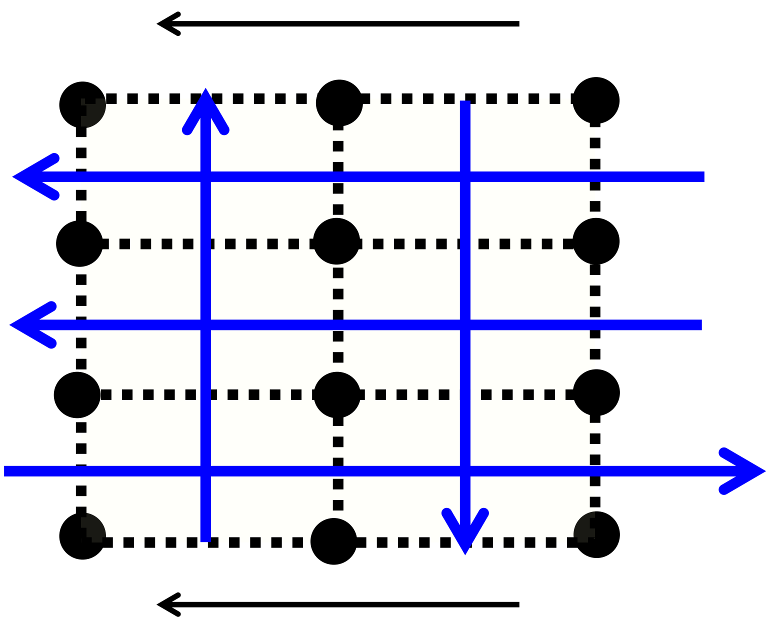
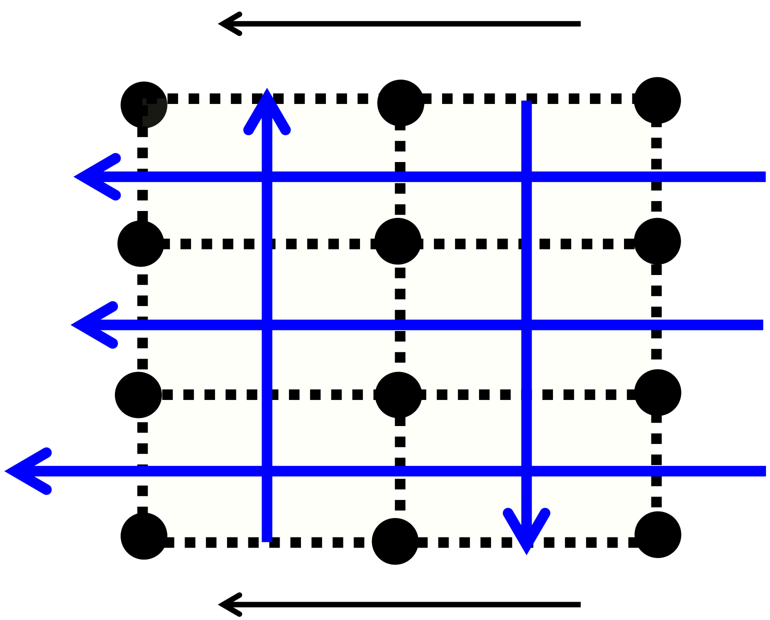
$\mathbb{Z}_3$ part of monodoromy

Suppose $\exists\,\Gamma$-sub. $\mathbb{Z}_3$-ori. guaranteeing H-face-2-coloring

$\forall$ component of the S-dual has two orientations





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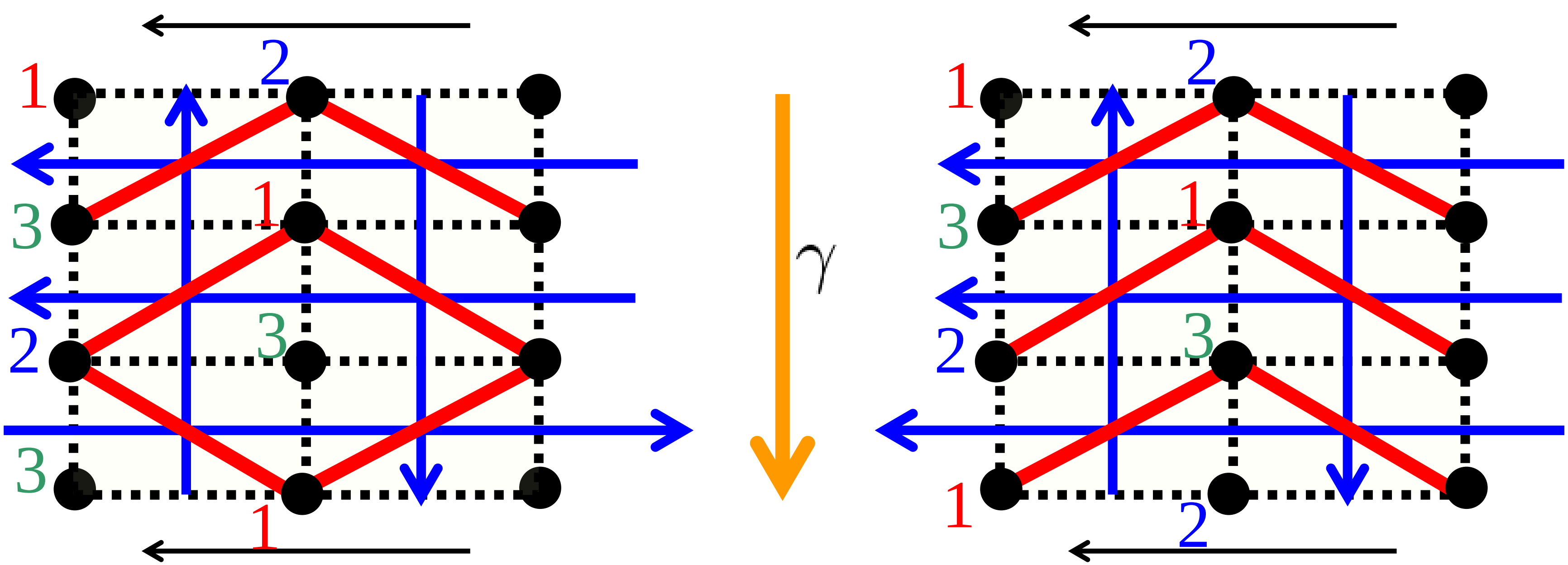

$$\sigma_T(\gamma) = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

$$\sigma_T(\gamma) = \mathrm{id}$$

$\mathbb{Z}_3$ part of monodoromy

Components of S-dual

``Crossing # between them''

$$\begin{array}{cc} & \begin{array}{cccc} P_1 & P_2 & \cdots & P_m \end{array} \\ \begin{array}{c} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_g \end{array} & \begin{pmatrix} u_{11} & u_{12} & \cdots & u_{1m} \\ u_{21} & u_{22} & \cdots & u_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ u_{g1} & u_{g2} & \cdots & u_{gm} \end{pmatrix} \end{array} \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} = \vec{0} \pmod 3$$

A base of the fundamental group of F^2

s.t. $\boxed{x_i \in \{1, 2\} \text{ for } \forall j}$

$\mathbb{Z}_3$ part of monodoromy

Thm :

G : mosaic of a surface

G has an extension to a 3-colorable triangulation

$\Leftrightarrow$

$\mathbb{Z}_2$ part

$\exists$ Γ -sub. $\mathbb{Z}_3$-ori. for

Γ : $\mathbb{Z}_2$-homologous to $\begin{cases} \text{an empty graph (orientable case)} \\ \text{an orientizing (nonorientable case)} \end{cases}$

and the previous system of equations has a solution.

$\mathbb{Z}_3$ part

Summary

Mosaics (all faces are tri- or quadrangular)

$$\begin{array}{ccc} \text{A mosaic } G & \longrightarrow & \text{3-colorable tri.} \\ \downarrow & & \uparrow \\ \text{Eulerian tri.} & \longrightarrow & \boxed{\begin{array}{c}\text{Monodoromy} \\ \mathbb{Z}_2 \text{ part} \longrightarrow \mathbb{Z}_3 \text{ part}\end{array}} \end{array}$$

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