Snarks that cannot be covered with four perfect matchings

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GGTW 2017, Ghent, August 2017

joint work with Martin Škoviera

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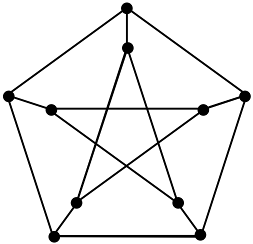

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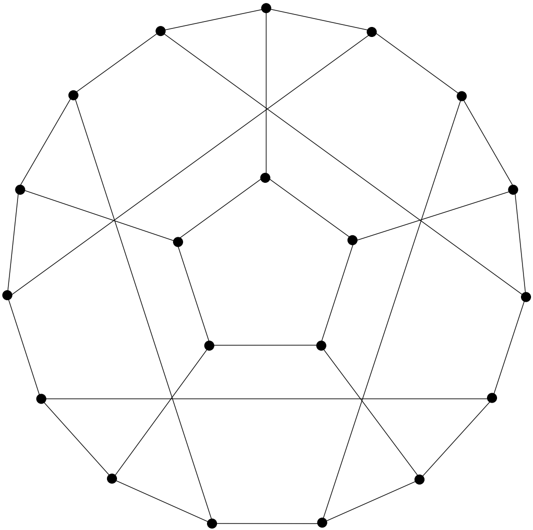
- it is an NP-complete problem to decide whether given cubic graph is snark or not [Holyer, 1981] (reduction from 3SAT)
- snarks are crucial for many conjectures and open problems (Cycle double cover conjecture, 5-Flow conjecture)

Perfect matchings in cubic graphs

Fulkerson Conjecture (Berge, Fulkerson, 1971)

Every bridgeless cubic graphs contains a family of six perfect matchings that together cover each edge exactly twice.

6 perfect matchings on I_5



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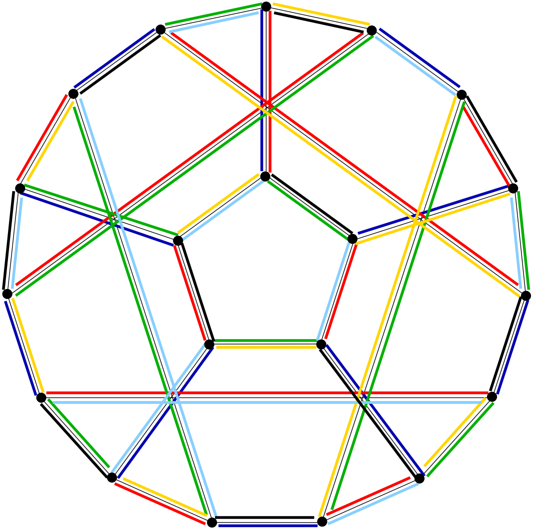

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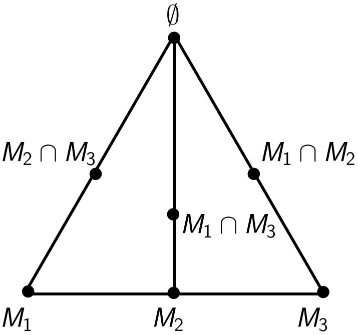
Theorem (Mazzuoccolo, 2011)

The Berge Conjecture and the Fulkerson Conjecture are equivalent.

Fan-Raspaud Conjecture

Fan-Raspaud Conjecture, 1994

Every bridgeless cubic graph has three perfect matchings with empty intersection.



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- Cubic graphs with $\tau(G) \leq 4$ are counterexamples to neither 5-CDCC nor Fan-Raspaud Conjecture

Point-line configurations

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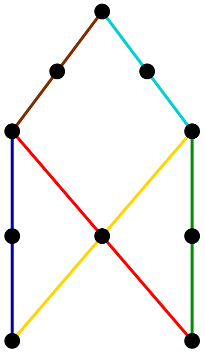
- configuration $\mathcal{C} = (P, B)$
  - P – finite set of points
  - B – finite set of blocks (3-element subsets of P such that for each pair of points of P there is at most one block of B which contains both of them)

Example: a configuration



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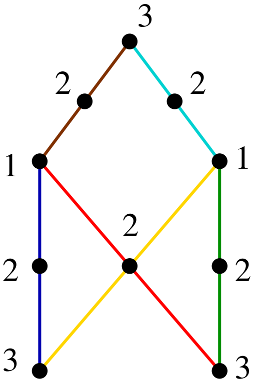

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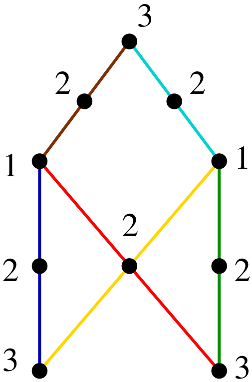
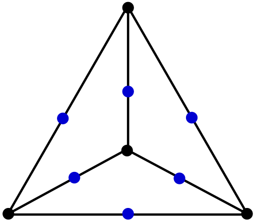


this configuration is not universal

"K_4"-configuration and four perfect matchings



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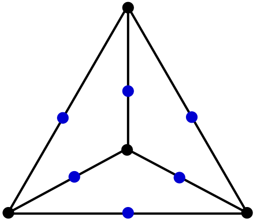

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- 10 points, 6 blocks
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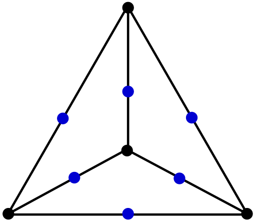

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Theorem (EM,Škoviera, 2017+)

A cubic graph G is $\mathcal{T}$-colourable $\Leftrightarrow$ the edges of G can be covered by at most 4 perfect matchings.

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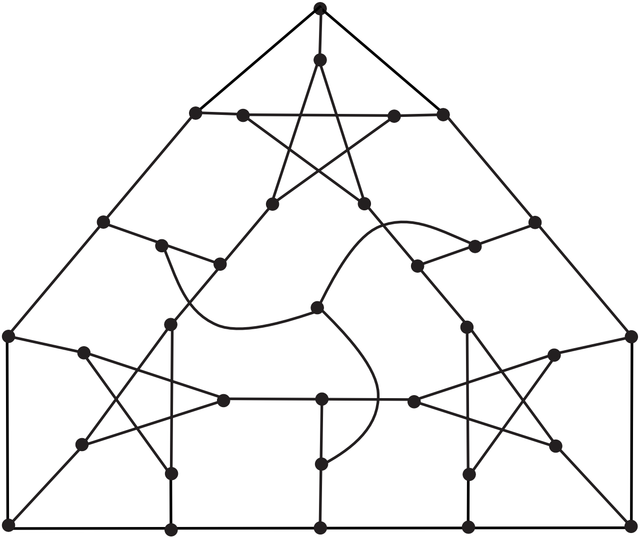
- both have $\tau(G) = 5$

A snark of order 34 with $\tau(G) = 5$



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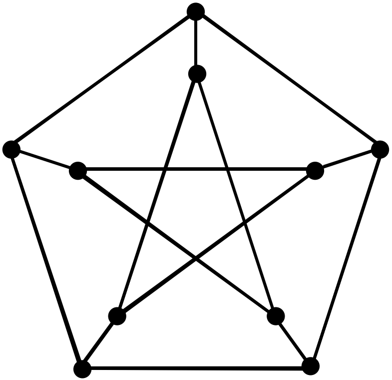

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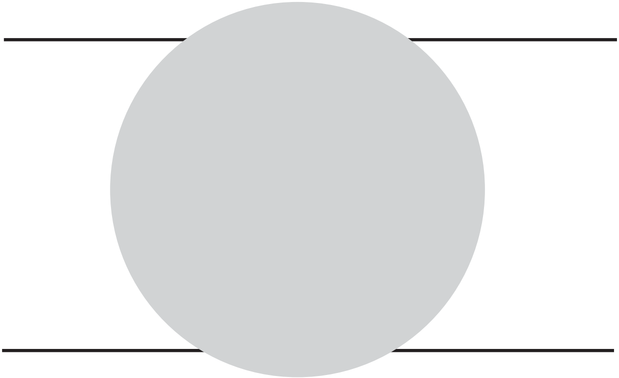

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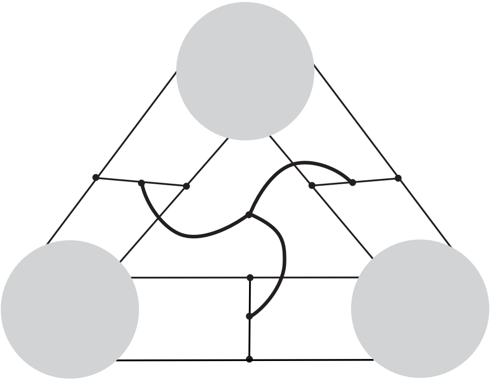

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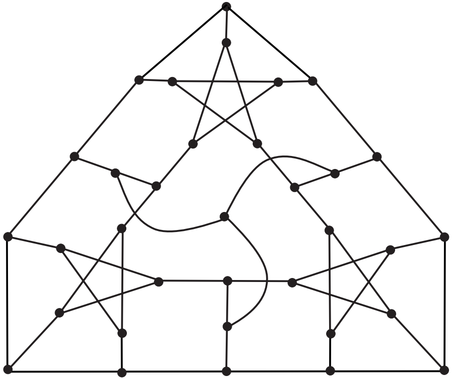

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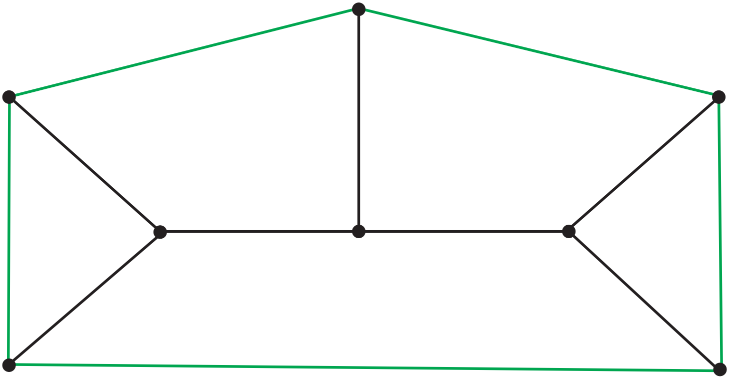

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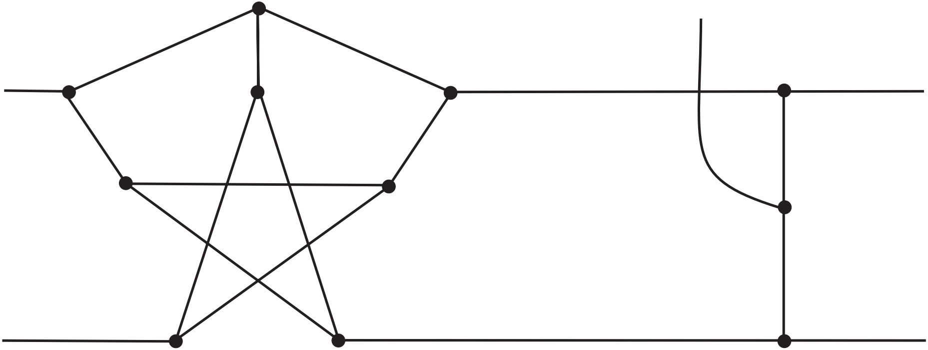

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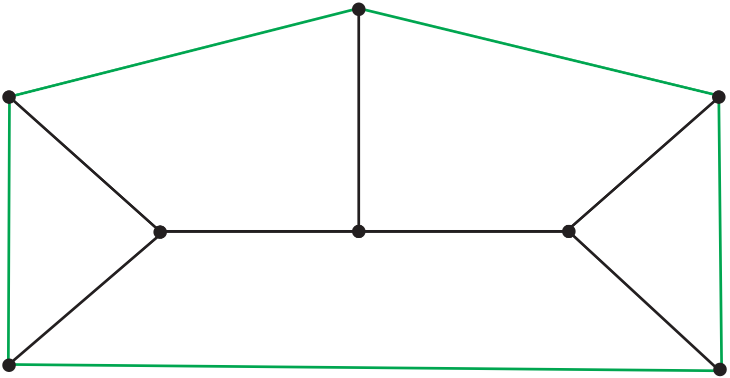

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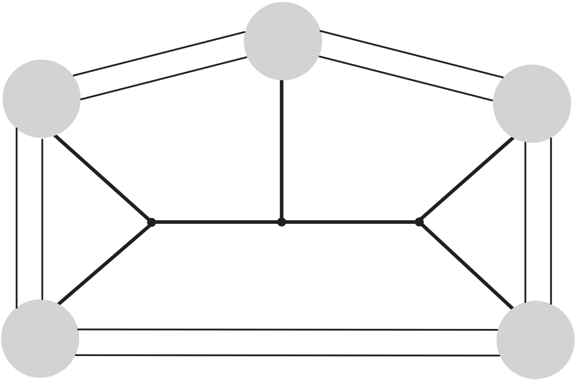

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However:

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- Proofs heavily depend on computer-aided arguments.

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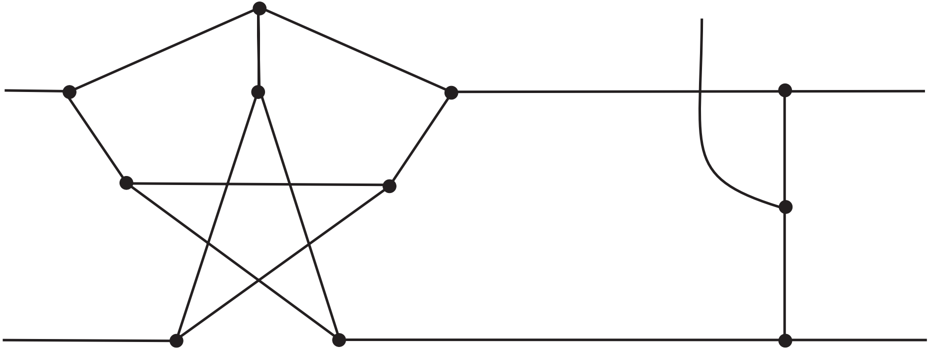

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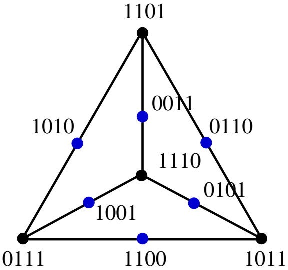
8.1 The pattern set of the Petersen fragment

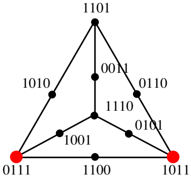
The pattern set of F_0 (42 patterns):

A A AB AC AD	A B CD AB AB	A BC D BC BC
A A AB C D	A B CD AC AC	A BC D BD BD
A AB A AC AD	A B CD C C	A BC D D D
A AB A BC BD	A B CD CD CD	AB AB AB AC AD
A AB AC A AD	A BC A AB BD	AB AC AB AB AD
A AB AC B BD	A BC B AB AD	AB AC AB BC CD
A AB AC C CD	A BC B BC CD	AB AC AD A A
A B AB AB CD	A BC BD A AB	AB AC AD AB AB
A B AB AC BD	A BC BD BC C	AB AC AD AD AD
A B AC A D	A BC BD BD D	AB AC AD B B
A B AC AB BD	A BC D A A	AB AC AD BC BC
A B C A AD	A BC D AB AB	AB AC AD BD BD
A B C C CD	A BC D AD AD	AB AC AD D D
A B CD A A	A BC D B B	AB CD AC AB BC

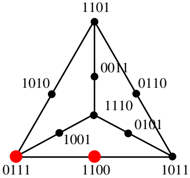
Tetrahedral $\mathbb{Z}_2^4$-flow



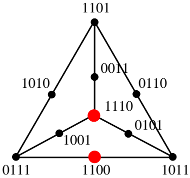
Types of connector inputs of size 2



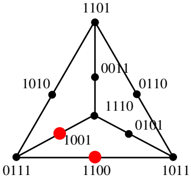
line–segment



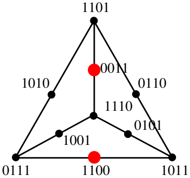
half–line



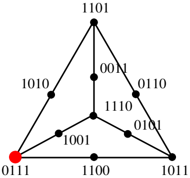
altitude



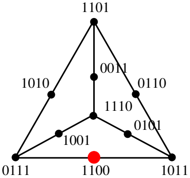
angle



axis



zero



zero

Set of transitions $\mathcal{M}$

Let $\mathcal{M}$ be the set of all transition through a $(2,2;1)$-pole containing all the transition of the following types:

- axis $\xrightarrow{1}$ half-line
- line-seg $\xrightarrow{1}$ half-line
- zero $\xrightarrow{1}$ half-line
- angle $\xrightarrow{1}$ half-line
- angle $\xrightarrow{1}$ altitude
- altitude $\xrightarrow{1}$ line-seg
- axis $\xrightarrow{2}$ line-seg
- line-seg $\xrightarrow{2}$ line-seg
- line-seg $\xrightarrow{2}$ zero
- zero $\xrightarrow{2}$ line-seg
- zero $\xrightarrow{2}$ angle
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- altitude $\xrightarrow{2}$ half-line

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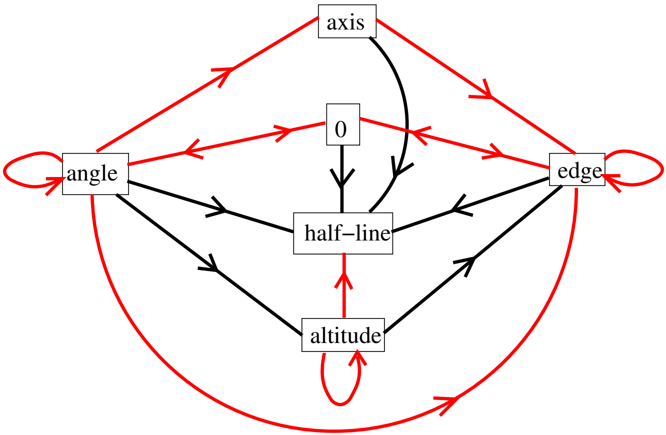

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Theorem (EM, Škoviera, 2017+)

- *Let S be a $(2,2)$-pole created from a snark G with $\tau(G) \geq 5$ by removing two adjacent vertices. Then $\mathbf{T}(S \circ I) \subseteq \mathcal{M}$.*

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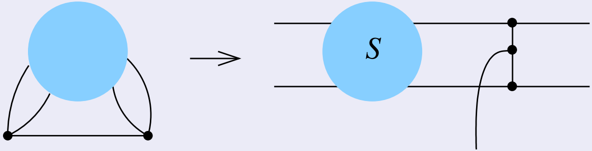

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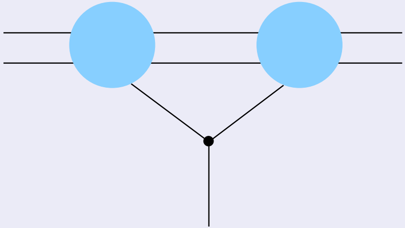

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- $\mathbf{T}(M_i) \subseteq \mathcal{M}$ *for $i \in \{1,2,3\}$. Then each transition of* $\mathbf{T}(\overline{M_1 \circ M_2 \circ M_3})$ *is of type angle $\to$ line-segment.*

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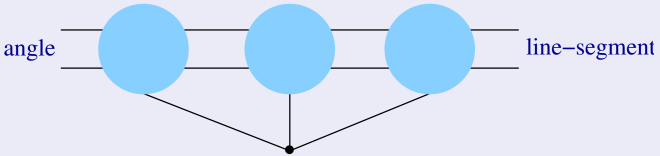

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- *For every even integer $n \geq 44$ there exists a snark G_n of order n with $\tau(G_n) \geq 5$.*
- *τ-resistance of a cubic graph can be arbitrarily high*

Thank you!