Recent progress towards a proof of the 4-4-4-Conjecture

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Theorem (Vizing)

Let G be a graph. Then $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$.

If $\chi'(G) = \Delta(G)$, then G is Class 1.

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Which **planar** graphs are Class 1 ?

- $\Delta(G) \geq 8 \implies \chi'(G) = \Delta$ (Vizing 1965).
- $\Delta(G) = 7 \implies \chi'(G) = 7$ (Sanders, Zhao 2001 ; Zhang 2000).
- $\Delta(G) \geq 5$, $g(G) \geq 4 \implies \chi'(G) = \Delta$ (Li, Luo 2003).
- $\Delta(G) \geq 4$, $g(G) \geq 5 \implies \chi'(G) = \Delta$ (Li, Luo 2003).

Δ	3	4	5	6	7	8
$g \geq 3$					✓	✓
$g \geq 4$			✓			
$g \geq 5$		✓				

Δ	3	4	5	6	7	8
$g \geq 3$	X	X	X		✓	✓
$g \geq 4$	X		✓			
$g \geq 5$	X	✓				

Δ	3	4	5	6	7	8
$g \geq 3$	X	X	X	?	✓	✓
$g \geq 4$	X	?	✓			
$g \geq 5$	X	✓				

Conjecture

Let G be a planar graph. If $\Delta(G) = 4$ and $g(G) \geq 4$, then $\chi'(G) = 4$.

Theorem (Li, Luo 2003)

Let G be a planar graph with $\Delta(G) = 4$ and $g(G) \geq 5$. Then G is 4-edge-colorable.

Properties of a minimal counter-example :

- Minimum degree is at least 2.
- If $d(v) = 2$, then $\forall u \in N(v)$, $d(u) = 4$.
- If $uv \in E(G)$ such that $d(u) = 2$ and $d(v) = 4$, then all neighbors of v but u are of degree 4.
- If $d(v) = 3$, then v has at least two neighbors of degree 4.

Theorem (Li, Luo 2003)

Let G be a planar graph with $\Delta(G) = 4$ and $g(G) \geq 5$. Then G is 4-edge-colorable.

Initial charge :

- $w(v) = \frac{3}{2}d(v) - 5$ for each $v \in V(G)$,
- $w(f) = d(f) - 5$ for each $f \in F(G)$.

The sum of initial charge of the whole graph is -10.

Theorem (Li, Luo 2003)

Let G be a planar graph with $\Delta(G) = 4$ and $g(G) \geq 5$. Then G is 4-edge-colorable.

Discharging rules :

(R1) Every 4-vertex sends 1 unit of charge to each 2-neighbor.

(R2) Every 4-vertex sends $1/4$ of charge to each 3-neighbor.

The sum of charge remains negative, but there is no vertex with negative charge after the discharging procedure.

To attack the conjecture, it is crucial to use planarity – $K_{4,4}^{+}$ is a graph with $\Delta = 4$, $g = 4$ which is not 4-edge-colorable.

The discharging similar to the case $\Delta = 4$ and $g = 5$ does not apply anymore : If $\Delta = 4$ and $g = 4$, then (a subgraph of) a square grid is charge-neutral.

- elements with negative charge : vertices of degree 2 and 3
- elements with positive charge : faces of size at least 5

Configurations

Let H be a graph with labels $\gamma : V(H) \to \mathbb{N}$ such that $\gamma(v) \geq d_H(v)$ for each $v \in V(H)$.
The graph H is a *configuration* in a graph G if there exists an isomorphism ξ from H to a subgraph X of G such that $\gamma(v) = d_G(\xi(v))$ for each $v \in V(H)$. We note

$$\partial_G(H) = \{e = uv \in E(G) : u \in V(H) \text{ and } v \in V(G \setminus H)\}.$$

the set of pending edges.

Reducible configuration

Let H be a configuration of G. We say that H is *reducible*, if there exists a smaller configuration H' with $\partial(H) = \partial(H')$, such that every 4-edge-coloring of $G' = (G \setminus H) \cup H'$ can be extended to a 4-edge-coloring of G after a finite sequence of Kempe chain switches.

Computing reducibility

Let H be a configuration and β be a coloration de $\partial(H)$. We say that β *extends to H* if there exists a coloring ψ of $H^* = H \cup \partial(H)$ such that $\psi|_{\partial(H)} = \beta$. Let

$$\Phi_0 = \{\beta : \exists \psi : E(H^*) \to \{1,2,3,4\} \mid \psi|_{\partial_G(H)} = \beta\}.$$

be the set of all colorings of pending edges that extend to H.

Computing reducibility

For $i \in \mathbb{N}_0$, let Φ_{i+1} be the set of all colorings of $\partial(H)$ such that there exists a pair of colors c, c' such that for every possible position of Kempe paths colored c and c' leaving H, for at least one of these its switching leads to a coloring in Φ_i. Let

$$\Phi(H) = \bigcup_{i \in \mathbb{N}} \Phi_i(H).$$

Computing reducibility

Let H and H' be two configurations with $\partial(H) = \partial(H')$.
Let G be a graph containing H, let $G' = (G \setminus H) \cup H'$.
If $\Phi_0(H') \subseteq \Phi(H)$, then for every 4-edge-coloration φ' of G' there exists a 4-edge-coloration φ of G.

We developped a tool for testing reducibility of configurations.

Input : Two configurations H and H' with the same set of pending edges.

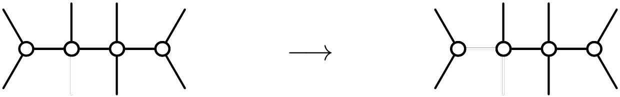
Output : $\Phi_0(H') \subseteq \Phi(H)$?

Re-confirmed reducible configurations :

$2-3 \qquad 2-4-3 \qquad 3-3-3$

New reducible configuration :

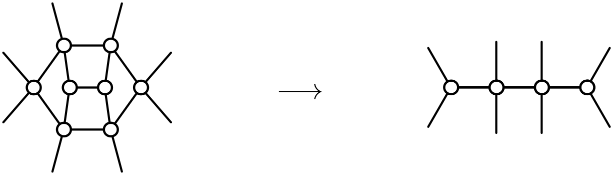
$3-3-4-3$



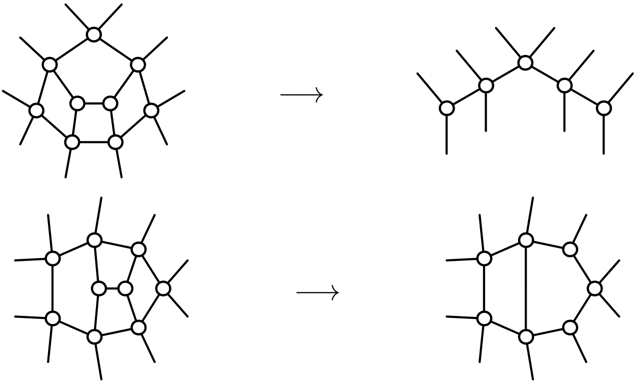
Not reducible by deleting a single edge :

$3-4-3-4-3 \qquad 3-3-4-4-3$

Two adjacent 3-vertices, surrounded by 4-faces :



Two adjacent 3-vertices, surrounded by three 4-faces :



What to do next :

- Find more reducible configurations, containing 3-vertices further from each other, and containing a 2-vertex ;
- Find a way to reduce small separators ;
- De-computerize the proofs by inventing a new specific language to manipulate the problem.

Thank you for your attention !