PROBLEMS

from the Problem Session of the

GHENT GRAPH THEORY WORKSHOP

on

LONGEST PATHS AND LONGEST CYCLES

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One would expect more edges to make the existence of Hamiltonian cycles more likely. Nevertheless, looking at what is known, polyhedra and the subset of triangulations (of the plane) seem to behave the same: though polyhedra can have as few as $2|V|$ edges while triangulations have $3|V| - 6$ edges, both are guaranteed to be Hamiltonian if they have up to three 3-cuts and can be non-hamiltonian if they have six 3-cuts. Whether four or five 3-cuts guarantee hamiltonicity is unknown for both classes.

Especially looking at numbers of Hamiltonian cycles (e.g. in 4-connected triangulations and polyhedra) one would expect a difference, as each edge in a 4-connected triangulation lies on several Hamiltonian cycles and – except for double wheels – in each 4-connected triangulation one can remove an edge from a triangulation and obtain a 4-connected polyhedron with fewer Hamiltonian cycles.

Nevertheless computational results show that for small vertex numbers the 4-connected polyhedra with the smallest number of Hamiltonian cycles have almost as many Hamiltonian cycles as the 4-connected triangulations with the smallest number of Hamiltonian cycles. And even more astonishing: for 18, 19 and 20 vertices one even gets the same optimal graphs for both classes: the double wheels. While for 18 vertices some other polyhedra have only few Hamiltonian cycles more than the double wheel, for 20 vertices there isn't even a polyhedron that comes close to the number of Hamiltonian cycles of the double wheel. So it looks like for $n \geq 18$ the number of Hamiltonian cycles of 4-connected polyhedra is the same value that was already conjectured for 4-connected triangulations by Hakimi, Schmeichel and Thomassen: $2|V|^2 - 12|V| + 16$ – the number of Hamiltonian cycles of the double wheel. While for 4-connected triangulations a linear lower bound for the number of Hamiltonian cycles has been proven, for 4-connected polyhedra only trivial constant bounds are known, though the real numbers seem to be the same...

Problems:

- *Determine for polyhedra and triangulations whether the existence of at most four or five* 3*-cuts guarantees the existence of Hamiltonian cycles.*
- *Prove nontrivial lower bounds on the number of Hamiltonian cycles in* 4*-connected polyhedra and maybe also in polyhedra with few* 3*-cuts.*

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For integers j and k with $4 \leq j \leq 5$ and $k \geq 1$, let e_k^j be the smallest integer l such that there is a j-connected plane triangulation G containing l prescribed edges of pairwise distance at least k such that there is no Hamiltonian cycle of G containing all these l edges. If e_k^j does not exist, then we write $e_k^j = \infty$.

Obviously, $e_k^j \leq e_{k+1}^j$ and $e_k^4 \leq e_k^5$.

In [F. Göring and J. Harant, Hamiltonian cycles through prescribed edges of at least 4-connected maximal planar graphs, *Discrete Math.* **310** (2010) 1491–1494], it is proved that $4 \leq e_1^5 \leq 9$ and $e_3^5 = \infty$.

Problems:

- *It is open whether e_2^5 and e_3^4 are finite or not.*
- *What are the exact values e_1^4 and e_1^5?*

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In [G. Y. Katona and H. Kierstead, Hamiltonian chains in hypergraphs, *J. Graph Theory* **30** (1999) 205–212] we gave a new definition for Hamiltonian cycles in uniform hypergraphs and proved a Dirac-type theorem. Since then the theorem was improved and generalized many different ways. A recent survey is [Y. Zhao, Recent advances on Dirac-type problems for hypergraphs, *The IMA Volumes in Mathematics and its Applications* **159** pp. 145–165]. However, all of the papers contain only Dirac-type results.

Problem:

Is there an Ore-type theorem?

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The following theorem characterizes edge-colored graphs containing no properly colored cycles.

Theorem (Grossman and Häggkvist, Yeo)**.** *Let G be an edge-colored graph containing no properly colored cycles. Then there is a vertex $z \in V(G)$ such that no component of $G - z$ is joint to z with edges of more than one color.*

So there is a sufficient condition for the existence of a properly colored cycle by connectivity and minimum color degree.

Corollary. *Let G be an edge-colored graph. If $\kappa(G) \geq 2$ and $\delta^c(G) \geq 2$, then G contains a properly colored cycle.*

Problem:

Do there exist functions $f(k)$ and $g(k)$ such that each edge-colored graph G satisfying $\kappa(G) \geq f(k)$ and $\delta^c(G) \geq g(k)$ contains k vertex-disjoint properly colored cycles?

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For a cycle C in a plane triangulation, triangles that share exactly i edges with C are said to be of *Type i (with respect to C)*, where $i \in \{0, 1, 2\}$. By Whitney's result, it is known that every 4-connected plane triangulation contains a Hamiltonian cycle. For possible applications (e.g. domination number, 2-walks of short length), we would like to find a Hamiltonian cycle with few triangles of Type 0 (which is equivalent to a Hamiltonian cycle with few triangles of Type 2). Precisely, I propose the following problem.

Problem:

Find the infimum c such that every 4-connected plane triangulation G contains a Hamiltonian cycle having at most $c|F(G)|$ triangles of Type 0, where $F(G)$ is the set of faces of G.

It would be also interesting to focus on only the interior of the cycle; i.e. a Hamiltonian cycle with few interior triangles of Type 0. Furthermore, the same is unknown for the 5-connected case. Those may have applications.

Note (*added after the workshop ended*). This problem has been solved by Gunnar Brinkmann (`gunnar.brinkmann@ugent.be`), Kenta Ozeki (`ozeki@nii.ac.jp`), and Nico Van Cleemput (`nicolas.vancleemput@ugent.be`). Please contact them for more details.

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Let G be a bridgeless cubic graph. Consider a list of k 1-factors of G. Let E_i be the set of edges contained in precisely i members of the k 1-factors. Let $\mu_k(G)$ be the smallest $|E_0|$ over all lists of k 1-factors of G.

It is known that if G is a cubic graph and $\mu_3(G) = 3$, then G has girth at most 6. Jaeger and Swart [Conjecture 1 and 2, in "Combinatorics 79" (eds.: Deza and Rosenberg), *Ann. Discrete Math.* **9** (1980) 305] conjectured that (1) the girth and (2) the cyclic connectivity of a snark is at most 6. The first conjecture is disproved by Kochol [Snarks without Small Cycles, *J. Combin. Theory Ser. B* **67** (1996) 34–47] and the second is still open. We believe that both statements of Jaeger and Swart are true for hypohamiltonian snarks.

Conjecture:

Let G be a snark. If G is hypohamiltonian, then $\mu_3(G) = 3$.

[E. Steffen, 1-Factor and Cycle Covers of Cubic Graphs, *J. Graph Theory* **78** (2015) 195–206]

Jan Goedgebeur (pers. communication) proved that the Conjecture is true for hypohamiltonian snarks with at most 36 vertices.

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Thomassen showed in 1978 that every planar hypohamiltonian graph contains a cubic vertex. Recently, Gunnar Brinkmann and I (see [arXiv:1606.01693]) showed that planar 3-connected graphs with at most three 3-cuts are Hamiltonian. By applying this theorem, I proved that every planar hypohamiltonian graph contains at least four cubic vertices. We know that there exists a planar hypohamiltonian graph (of order 40) containing 30 cubic vertices. No planar hypohamiltonian graph with fewer cubic vertices is known.

Problem:

Prove that planar hypohamiltonian graphs contain more than four cubic vertices, or find a planar hypohamiltonian graph with fewer than 30 cubic vertices.