Finding spanning trees with few leaves using DFS

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MŰEGYETEM 1782

Ghent Graph Theory Workshop, 2016.08.01.

Introduction

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Definition

- The *minimum leaf number* $\mathrm{ml}(G)$ is the minimum number of leaves (vertices of degree 1) of the spanning trees of G.
- The *minimum branch number* $\mathrm{s}(G)$ is the minimum number of branches (vertices of degree at least 3) of the spanning trees of G.
- A tree T is a *k-tree* if all vertices have degree at most k.
- The minimum sum of degrees of k independent vertices of G is denoted by $\delta_k(G)$.

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Theorem (Kano-Kyaw-Matsuda-Ozeki-Saito-Yamashita, 2012)

If $\delta_{k+1}(G) \geq n - k$, then G has a spanning 3-tree with at most k leaves.

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Every connected claw-free graph G has a DFS-tree T, such that no two of the d-leaves of T have a common neighbour. Moreover, if v is not a cut vertex of G, then T can be chosen such that it is rooted at v.

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- A local improvement step leads to T', a spanning 3-tree with $\leq k$ leaves.

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- Let T be a DFS-tree of G with a minimum number of d-leaves
- and among these with a minimum length sum of d-leaves (length of a leaf ℓ: length of the path between ℓ and the closest branch to ℓ in T).

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- Then T possess the desired property.

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- We obtain that $(a, x') \in E(G)$ or $(b, x') \in E(G)$, contradicting the choice of T.
- If $x = r$ and x has 1 child, let the child be x' and consider $G[\{x, x', a, b\}]$.
- If $x = r$ and x has at least 2 children, let us place a above r in T.

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Theorem (Ainouche-Broersma-Veldman, 1990)

Every connected claw-free graph of diameter at most 2 is traceable.

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If $\alpha(G^2) \leq m+k-1$, then $\mathrm{ml}(G) \leq k$.

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If $\alpha(G^2) \leq m + k - 1$, then $\mathrm{ml}(G) \leq k$.

$$\delta_{k+1}(G) \geq n - k \;\Rightarrow\; \alpha(G^2) \leq k \;\Rightarrow\; \mathrm{ml}(G) \leq k - m + 1.$$

Last slide before lunch

Thank you!