# Connections between decomposition trees of 3-connected plane triangulations and hamiltonian properties

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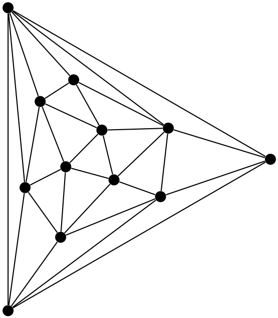
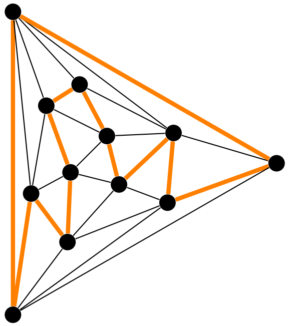
# Triangulation

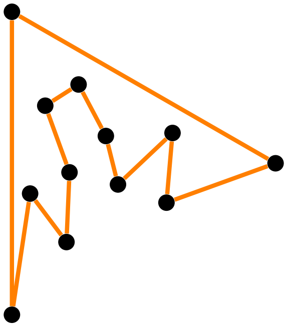
A triangulation is a plane graph in which each face is a triangle.



# Hamiltonian cycle

A hamiltonian cycle in $G(V, E)$ is a subgraph of $G(V, E)$ which is isomorphic to $C_{|V|}$.

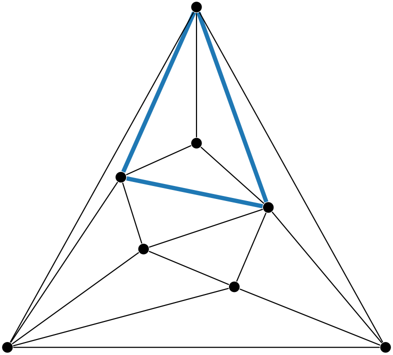




A graph is hamiltonian if it contains a hamiltonian cycle.

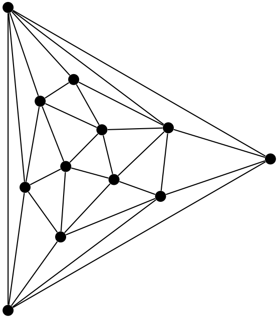
# Separating triangles

A separating triangle $S$ in a triangulation $T$ is a subgraph of $T$ such that $S$ is isomorphic to $C_3$ and $T - S$ has two components.



# 4-connected triangulations

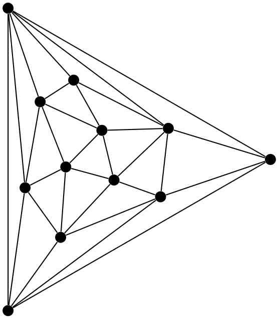
A triangulation is 4-connected if and only if it contains no separating triangles.



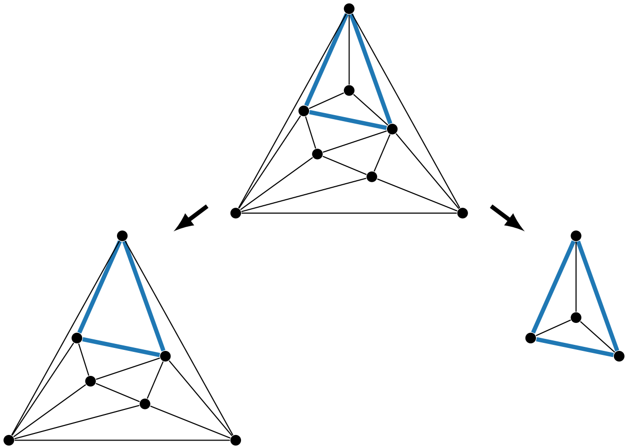
# Whitney

**Theorem (Whitney, 1931)**

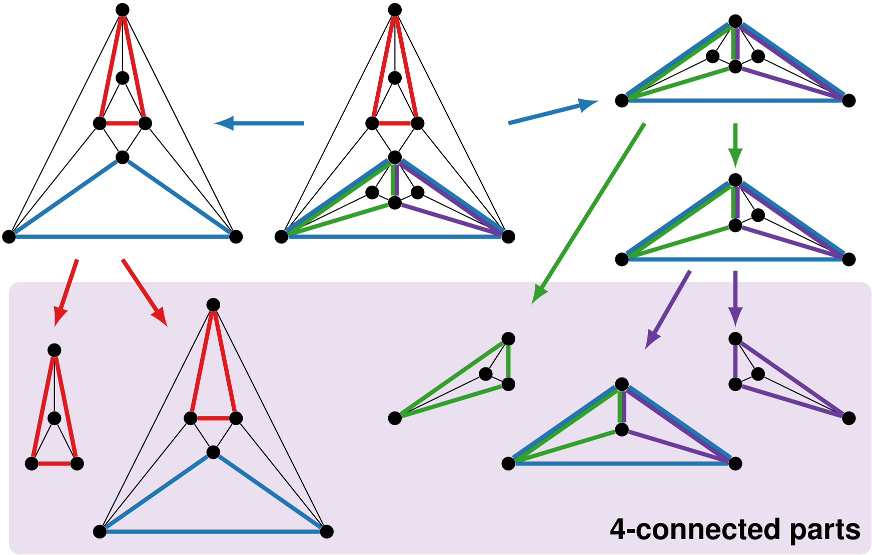
*Each triangulation without separating triangles is hamiltonian.*



# Splitting triangulations

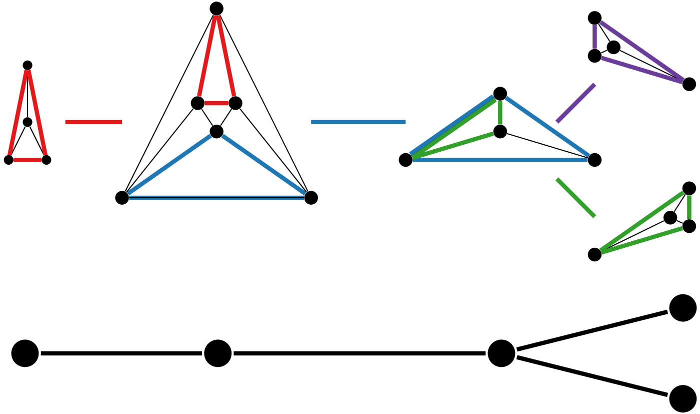


# Recursively splitting triangulations



# Decomposition tree

Vertices: 4-connected parts
Edges: separating triangles



# Decomposition trees and hamiltonicity

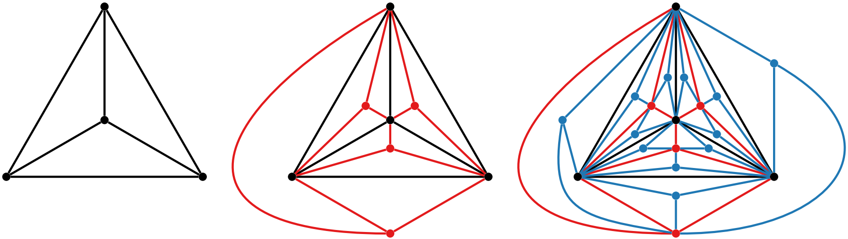
For each tree $T$ there exist hamiltonian triangulations which have $T$ as decomposition tree.

A triangulation $G$ with decomposition tree $T$ is hamiltonian if ...

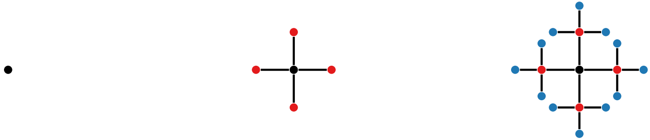
- Whitney (1930): $|E(T)| = 0$
- Thomassen (1978), Chen (2003): $|E(T)| \leq 1$
- Böhme, Harant, Tkáč (1993): $|E(T)| \leq 2$
- Jackson, Yu (2002): $\Delta(T) \leq 3$

# Jackson and Yu

$\Delta(T) \leq 4$ is not sufficient to imply hamiltonicity.



hamiltonian hamiltonian not hamiltonian

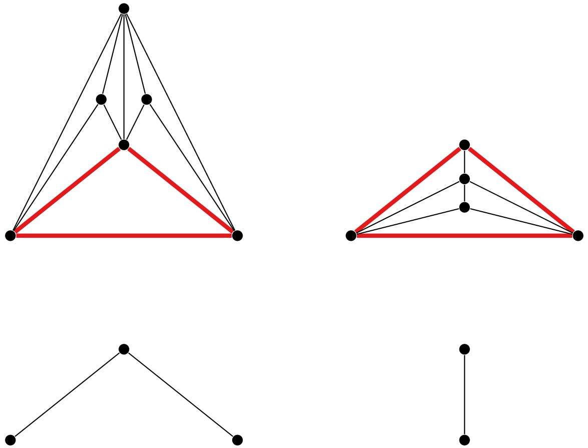


# Question

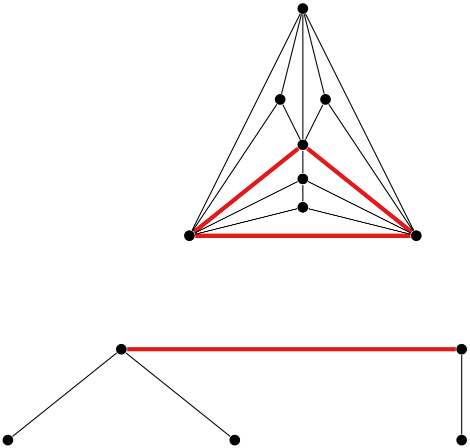
Can the result of Jackson and Yu be improved?

Which trees can arise as decomposition trees of non-hamiltonian triangulations?

# Subdividing a face with a graph



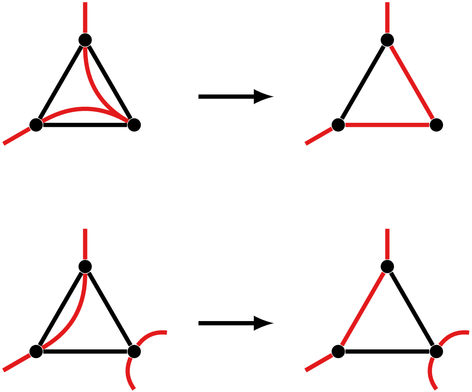
# Subdividing a face with a graph



# Subdividing a non-hamiltonian triangulation

## Lemma

*When a non-hamiltonian triangulation is subdivided, then the resulting graph is also non-hamiltonian.*

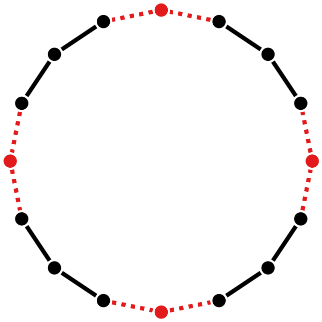


# Toughness

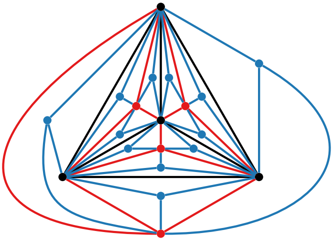
A graph is 1-tough if it cannot be split into $k$ components by removing less than $k$ vertices.

# Toughness

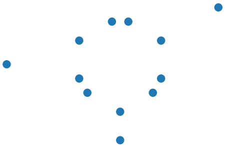
A hamiltonian graph is 1-tough.



Graphs that are not 1-tough are trivially non-hamiltonian.



Remove 4 black and 4 red vertices

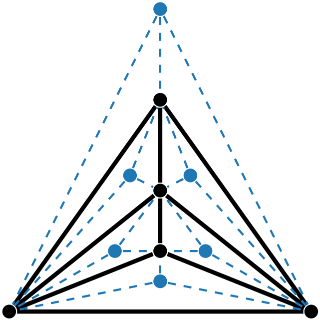


12 blue components remain

# Creating a non-hamiltonian plane graph

**Lemma**

*When in a plane graph with more faces than vertices each face is subdivided, then the resulting plane graph is non-hamiltonian.*



The subdivided graph is not 1-tough.

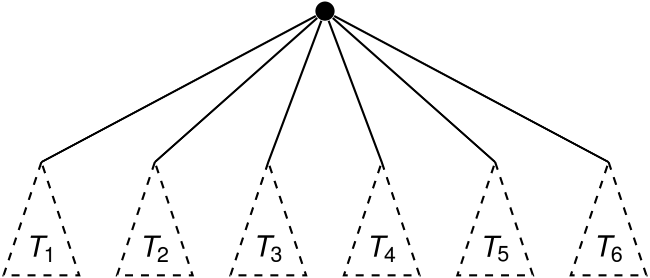
# Decomposition trees with $\Delta \geq 6$

**Theorem**

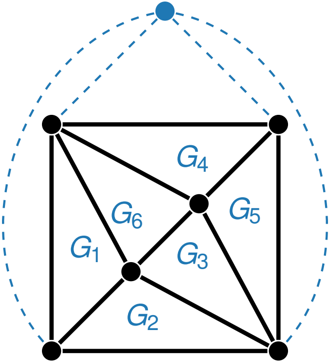
*For each tree $T$ with $\Delta(T) \geq 6$, there exists a non-hamiltonian triangulation $G$, such that $T$ is the decomposition tree of $G$.*

Constructive proof.

Assume $\Delta(T) = 6$.

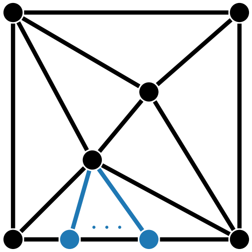



Choose triangulation $G_i$ with decomposition tree $T_i$ $(1 \leq i \leq 6)$




A non-hamiltonian triangulation with $T$ as decomposition tree.

$\Delta(T) > 6$



# Remaining cases

$$\Delta : \quad 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad \cdots$$

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Not the decomposition tree of non-hamiltonian triangulation

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$$\Delta : \; 0 \;\; 1 \;\; 2 \;\; 3 \;\; 4 \;\; 5 \;\; 6 \;\; 7 \;\; \cdots$$

Not the decomposition tree of non-hamiltonian triangulation

Possibly the decomposition tree of non-hamiltonian triangulation

# Remaining cases

$$\Delta : \quad \underbrace{0 \quad 1 \quad 2 \quad 3}_{} \quad \underbrace{4 \quad 5}_{} \quad \underbrace{6 \quad 7 \quad \cdots}_{}$$

Not the decomposition tree of non-hamiltonian triangulation

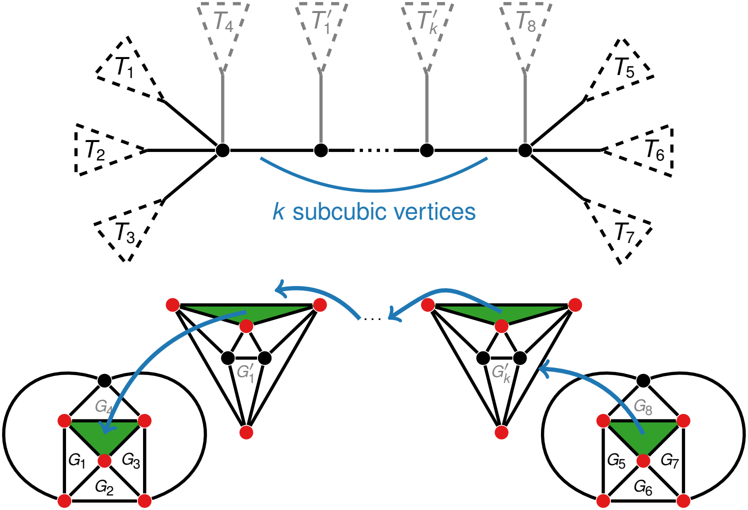
?

Possibly the decomposition tree of non-hamiltonian triangulation

# Multiple degrees $> 3$

**Theorem**

*For each tree $T$ with at least two vertices with degree $> 3$, there exists a non-hamiltonian triangulation $G$, such that $T$ is the decomposition tree of $G$.*




red vertices: $5 + k + (5 - 3) = 7 + k$

components: $4 + k + 4 = 8 + k$

Remaining cases: trees with one vertex of degree 4 or 5 and all other degrees at most 3.

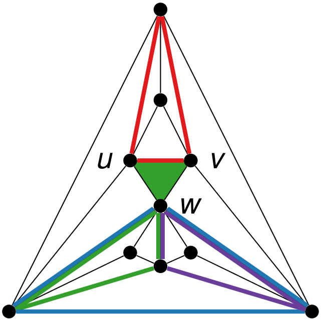
# One vertex of degree 4 or 5

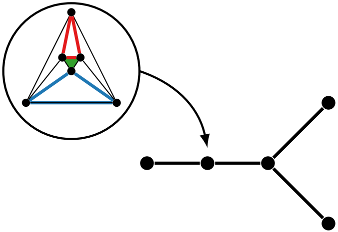
**Theorem**

*Let $G$ be a triangulation with decomposition tree $T$ with only one vertex of degree 4 or 5 and all other vertices of degree at most 3. Then $G$ is 1-tough.*

**Theorem (Jackson and Yu, 2002)**

*Let $G$ be a triangulation with decomposition tree $T$, $\Delta(T) \leq 3$ and $uvw$ a facial triangle of $G$ that is also a facial triangle in a vertex of $T$ with degree at most 2.*
*Then $G$ has a hamiltonian cycle through $uv$ and $vw$.*



# This implies:

Non-hamiltonian triangulations with decomposition trees with one vertex of degree $k \geq 4$ and all others of degree at most 3 exists if and only if...

- non-hamiltonian triangulations with decomposition tree $K_{1,k}$ exist.
- 4-connected triangulations exist with facial triangles $t_1, \ldots, t_k$ so that no hamiltonian cycle $C$ and distinct edges $e_1, \ldots, e_k \in C$ exist such that $e_i \in t_i$.

Also valid for $k \in \{1, 2, 3\}$

# Specialised search

**Lemma**

*All triangulations on at most 31 vertices with $K_{1,4}$ as decomposition tree are hamiltonian.*

**Lemma**

*All triangulations on at most 27 vertices with $K_{1,5}$ as decomposition tree are hamiltonian.*

# ... and now?

Prove that for each 4-tuple of vertex-disjoint triangles in a 4-connected triangulation there exists a hamiltonian cycle that shares an edge with each of the triangles.

or

Find a counterexample.

# ... and now?

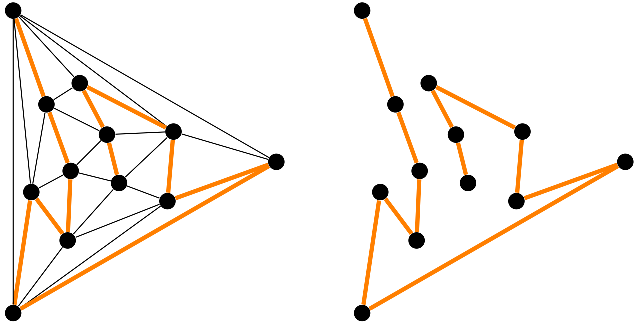
Prove that for each 5-tuple of triangles $T_1, T_2, T_3, T_4, T_5$ in a 4-connected triangulation there exists a hamiltonian cycle $C$ and distinct edges $e_1, e_2, e_3, e_4, e_5 \in C$ such that $e_i \in T_i$.

or

Find a counterexample.

# Hamiltonian path

A hamiltonian path in $G(V, E)$ is a subgraph of $G(V, E)$ which is isomorphic to $P_{|V|}$.



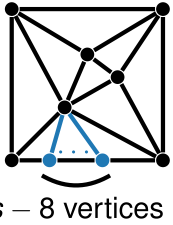
# Traceable

A graph $G(V, E)$ is traceable if it contains a hamiltonian path.

# Decomposition trees with $\Delta \geq 8$

**Theorem**

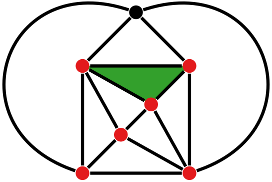
*For each tree $T$ with $\Delta(T) \geq 8$, there exists a non-traceable triangulation $G$, such that $T$ is the decomposition tree of $G$.*



# Decomposition trees with $\Delta \in \{6, 7\}$

## Theorem

*For each tree $T$ with a pair of vertices with degrees $k_1$ and $k_2$ with $(k_1, k_2) \in \{(6,4),(6,5),(6,6),(7,4),(7,5),(7,6),(7,7)\}$ and all others of degree at most 3, there exists a non-traceable triangulation $G$, such that $T$ is the decomposition tree of $G$.*

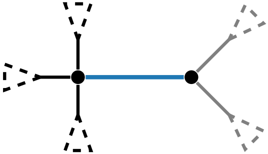


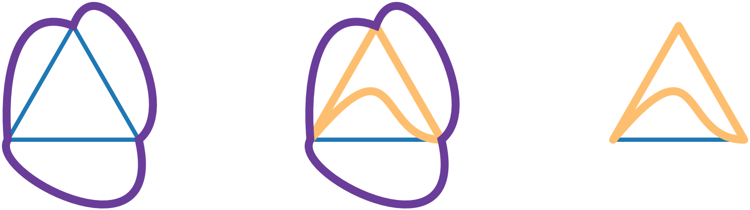
# Decomposition trees with $\Delta = 4$

## Theorem

*Let $T$ be a tree with one vertex of degree 4 and all others of degree at most 3. Then any triangulation which has $T$ as decomposition tree is traceable.*

# Decomposition trees with $\Delta = 4$





# Hamiltonian-connected

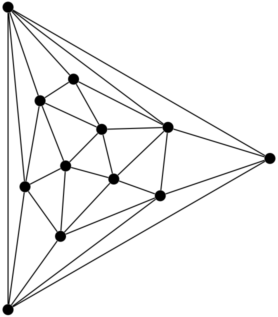
A hamiltonian path connecting $x$ and $y$ is a hamiltonian path $P$ such that $x$ and $y$ have degree 1 in $P$.

A graph $G(V, E)$ is hamiltonian-connected if for each pair $x, y$ of distinct vertices in $V$ there exists a hamiltonian path connecting $x$ and $y$.

# 4-connected triangulations

**Theorem (Thomassen, 1983)**

*Each triangulation without separating triangles is hamiltonian-connected.*



# 3-connected triangulations

**Theorem**

*Let $G$ be a 3-connected triangulation such that there is an edge $e$ which is contained in all separating triangles. Then $G$ is hamiltonian-connected.*

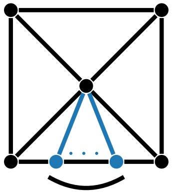
# Decomposition tree

**Theorem**

*Let $T$ be a tree with maximum degree 1. Then any triangulation which has $T$ as decomposition tree is hamiltonian-connected.*

**Theorem**

*Let $T$ be a tree with maximum degree at least 4. Then $T$ is the decomposition tree of a 3-connected triangulation which is not hamiltonian-connected.*



$\Delta - 4$ vertices

# Decomposition tree

**Lemma**

*On up to 21 vertices all triangulations that have a decomposition tree with maximum degree 2 are all hamiltonian-connected.*

**Lemma**

*On up to 20 vertices all triangulations that have a decomposition tree with maximum degree 3 are all hamiltonian-connected.*

# Summary for hamiltonian-connectedness

$$\Delta : \quad 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad \cdots$$

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$$\Delta : \quad 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad \cdots$$

Always hamiltonian-connected

# Summary for hamiltonian-connectedness

$$\Delta : \quad \underbrace{0 \quad 1}_{\text{Always hamiltonian-connected}} \quad 2 \quad 3 \quad \underbrace{4 \quad 5 \quad 6 \quad 7 \quad \cdots}_{\text{Possibly not hamiltonian-connected}}$$

# Summary for hamiltonian-connectedness

$\Delta$ : 0 1 2 3 4 5 6 7 $\cdots$

Always hamiltonian-connected

?

Possibly not hamiltonian-connected

# Overview

**Definition**

A graph $G$ is $k$-edge-hamiltonian-connected if for any $X \subset \{x_1x_2 : x_1, x_2 \in V(G), x_1 \neq x_2\}$ such that $1 \leq |X| \leq k$ and $X$ is a forest of paths, $G \cup X$ has a hamiltonian cycle containing all edges in $X$.

1-edge-hamiltonian-connected is equivalent to hamiltonian-connected.

**Definition**

A graph $G$ is $k$-hamiltonian if for any $k$ vertices $v_1, \ldots, v_k$ in $G$, $G - \{v_1, \ldots, v_k\}$ is hamiltonian.

# Table

| | Traceable | Hamiltonian | Hamiltonian-connected | 2-edge-hamiltonian | 1-hamiltonian | 2-hamiltonian |
|---|---|---|---|---|---|---|
| 0 | | | | | | |
| 1,1 | | | | | | |
| 2,... | | | | | | |
| 3,... | | | | | | |
| 4,(≤ 3),... | | | | | | |
| 4,4,... | | | | | | |
| 5,(≤ 3),... | | | | | | |
| 5,4,... | | | | | | |
| 5,5,... | | | | | | |
| 6,(≤ 3),... | | | | | | |
| 6,4,... | | | | | | |
| 6,5,... | | | | | | |
| 6,6,... | | | | | | |
| 7,(≤ 3),... | | | | | | |
| 7,4,... | | | | | | |
| 7,5,... | | | | | | |
| 7,6,... | | | | | | |
| 7,7,... | | | | | | |
| 8,... | | | | | | |