

Complexity questions for minimally t -tough graphs

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Introduction

Definition

Let t be a positive real number. A graph G is called **t -tough**, if $\omega(G - S) \leq |S|/t$ for any cutset S of G . The **toughness** of G , denoted by $\tau(G)$, is the largest t for which G is t -tough, taking $\tau(K_n) = \infty$ for all $n \geq 1$.

Definition

A graph G is said to be **minimally t -tough**, if $\tau(G) = t$ and $\tau(G - e) < t$ for all $e \in E(G)$.

Conjecture (Kriesell)

Every minimally 1-tough graph has a vertex of degree 2.

Conjecture (Generalized Kriesell)

Every minimally t -tough graph has a vertex of degree $\lceil 2t \rceil$.

Motivation

Theorem

If G is a minimally 1-tough claw-free graph of order n , then $G = C_n$.

\implies The class of minimally 1-tough claw-free graphs is **not** rich.

Question

Is this the case for other special classes of minimally 1-tough graphs?

Question

Is this the case for the whole class of minimally 1-tough graphs?



Special Graph Classes



Constructing minimally t -tough graphs

- Start with any graph, compute its toughness: t
- Try to remove an edge.
 - ▶ If the toughness decreases then put back the edge, and try another one.
 - ▶ If the toughness does not decrease, then remove the edge permanently.
- If no edge can be removed without decreasing the toughness, then the remaining graph is minimally t -tough.

This is probably not a polynomial time algorithm, since computing the toughness is difficult.

Theorem (Bauer, Hakimi, Schmeichel, 1990)

For any positive rational number t , t -TOUGH is NP-hard.

The algorithm does not work in special graph classes, if the graph class is not monotone.



Chordal graphs

Definition

A graph is *chordal* if it does not contain an induced cycle of length at least 4.

Theorem

For any rational number $\frac{1}{2} < t \leq 1$, there exist no minimally t -tough chordal graphs.

$t \leq \frac{1}{2}$: A tree T is always a minimally $\frac{1}{\Delta(T)}$ -tough chordal graph.

Question

Can we characterise minimally t -tough chordal graphs if $t \leq \frac{1}{2}$?

Chordal graphs

The Generalized Kriesell Conjecture is true in this case.

Theorem

Let $t \leq \frac{1}{2}$ be a positive rational number, G a minimally t -tough chordal graph. Then G has a vertex of degree 1.

Question

What about minimally t -tough chordal graphs if $t > 1$?



Split graphs

Definition

A graph is a *split* graph, if its vertices can be partitioned into a clique and an independent set.

A split graph is always chordal.

Theorem (Kratsch, Lehel, Müller, 1996; Woeginger, 1998)

For any rational number $t \geq 0$, the class of t -tough split graphs can be recognized in polynomial time.



Split graphs

Theorem

For any rational number $t > \frac{1}{2}$, there exist no minimally t -tough split graphs.

Theorem

Let $t \leq \frac{1}{2}$ be an arbitrary rational number and G a minimally t -tough split graph, partitioned into a clique C and an independent set I . Then there exists a positive integer b , for which $t = \frac{1}{b}$ and $|C| \leq 3$. Moreover,

- 1 either G is a tree with at most two internal vertices and with $\Delta(G) = b$,*
- 2 or $|C| = 3$, every vertex in I has degree 1 and every vertex in C has degree $b + 1$.*

Split graphs

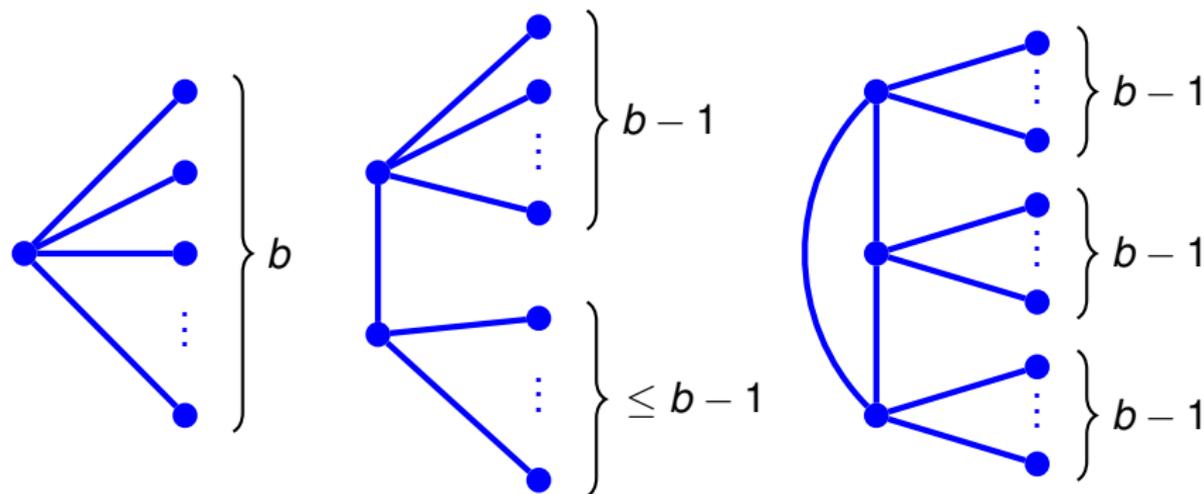


Figure : All minimally $\frac{1}{b}$ -tough split graphs.

The Generalized Kriesell Conjecture is true in this case.

Corollary

For any integer $b \geq 2$, every minimally $\frac{1}{b}$ -tough split graph has a vertex of degree 1.

Claw-free graphs

Definition

The graph $K_{1,3}$ is called a **claw**. A graph is said to be **claw-free**, if it does not contain a claw as an induced subgraph.

Theorem (Matthews, Sumner, 1984)

If G is a noncomplete claw-free graph, then $2\tau(G) = \kappa(G)$.

Since $\kappa(G)$ is an integer, $\tau(G)$ is an integer or half of an integer:

$\frac{1}{2}, 1, \frac{3}{2}, 2, \dots$

Corollary

For any rational number $t \geq 0$, the class of t -tough claw-free graphs can be recognized in polynomial time.



Claw-free graphs

Theorem

If G is a minimally 1-tough claw-free graph of order n , then $G = C_n$.

Kriesell Conjecture is true.

Theorem

The class of minimally $\frac{1}{2}$ -tough claw-free graphs can be recognized in polynomial time.

Lemma

If G is a minimally $\frac{1}{2}$ -tough claw-free graph, then

- *all of its cycles have length 3,*
- *each vertex of each triangle is a cut vertex,*
- *if it is a tree, then $\Delta(G) \leq 2$.*

Claw-free graphs

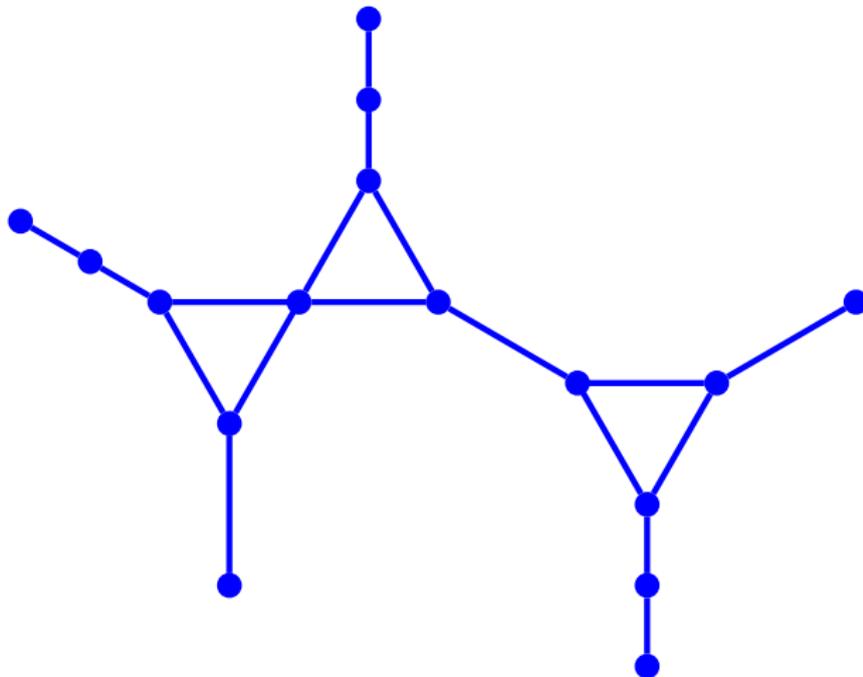


Figure : A minimally $\frac{1}{2}$ -tough claw-free graph.



Claw-free graphs

Question

What about minimally t -tough claw-free graphs if $t > 1$?



$2K_2$ -free graphs

Definition

A graph is said to be $2K_2$ -free, if it does not contain an independent pair of edges as an induced subgraph.

Theorem (Broersma, Patel, Pyatkin, 2014)

For any rational number $t \geq 0$, the class of t -tough $2K_2$ -free graphs can be recognized in polynomial time.

Theorem

For any positive rational number t , the class of minimally t -tough $2K_2$ -free graphs can be recognized in polynomial time.

$2K_2$ -free graphs

Claim

The graphs C_4 and C_5 are minimally 1-tough $2K_2$ -free graphs.

Claim

For every positive integer b , the graph $K_{1,b}$ is a minimally $\frac{1}{b}$ -tough $2K_2$ -free graph.

Question

*Do minimally t -tough $2K_2$ -free graphs exist for other t values?
Is the Generalized Kriesell Conjecture true in this case?*



The Whole Class of Minimally t -tough Graphs

joint work with István Kovács and Kitti Varga



The Class of Minimally t -tough Graphs

Question

Is it a rich graph class?

If it is difficult to decide if a graph is minimally t -tough, then yes.

Is it in NP? Is it in coNP? It is unknown. 🤔

The complexity class DP

Definition (Papadimitriou, Yannakakis, 1984)

A language L is in the **class DP** if there exist two languages $L_1 \in NP$ and $L_2 \in coNP$ such that $L = L_1 \cap L_2$.

Claim

For every positive rational number t , $MIN-t-TOUGH \in DP$.

Proof.

Let

$$L_1 = \{G \text{ graph} \mid \forall e \in E(G) : G - e \text{ is not } 1\text{-tough}\}$$

and

$$L_2 = \{G \text{ graph} \mid G \text{ is } 1\text{-tough}\}.$$

Then $L_1 \in NP$, $L_2 \in coNP$ and $MIN-t-TOUGH = L_1 \cap L_2$. □

The complexity class DP

Claim

$$NP \cup coNP \subseteq DP$$

Proof.

- If $L \in NP$ then $L = L \cap \Sigma^*$, so $L \in DP$, since $\Sigma^* \in coNP$.
- If $L \in coNP$ then $L = L \cap \Sigma^*$, so $L \in DP$, since $\Sigma^* \in NP$.



The complexity class DP

Definition

A language L is **DP-complete** if $L \in DP$ and all $L' \in DP$ have polynomial time reduction to L .

DP-complete problems:

- SAT-UNSAT
- UNIQUE-SAT
- CRITICAL-SAT
- CRITICAL-3-COLOR
- EXACT-4-COLOR
- MAX-NON-HAMILTONIAN
- EXACT-LONGEST-CYCLE
- EXACT-TSP
- CRITICAL-CLIQUE
(Papadimitriou, Wolfe, 1988)
- α -CRITICAL ($\alpha(G) < k$ and $\forall e \in E(G) : \alpha(G - e) \geq k$)

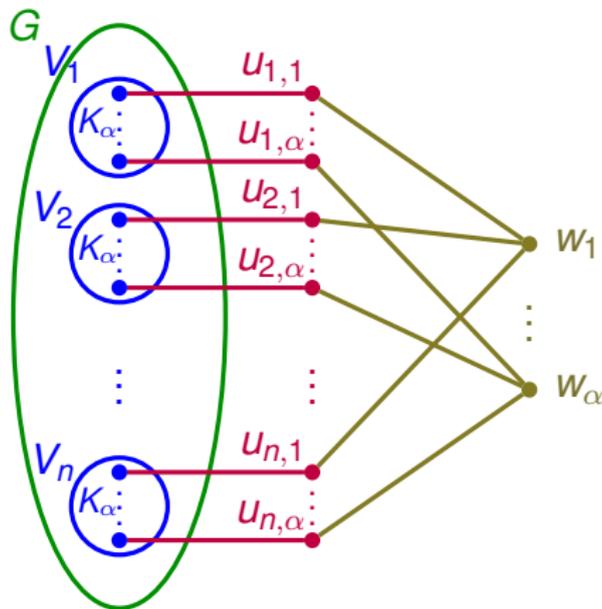


New results

Theorem

MIN-1-TOUGH is DP-complete.

α -CRITICAL \leq_p MIN-1-TOUGH

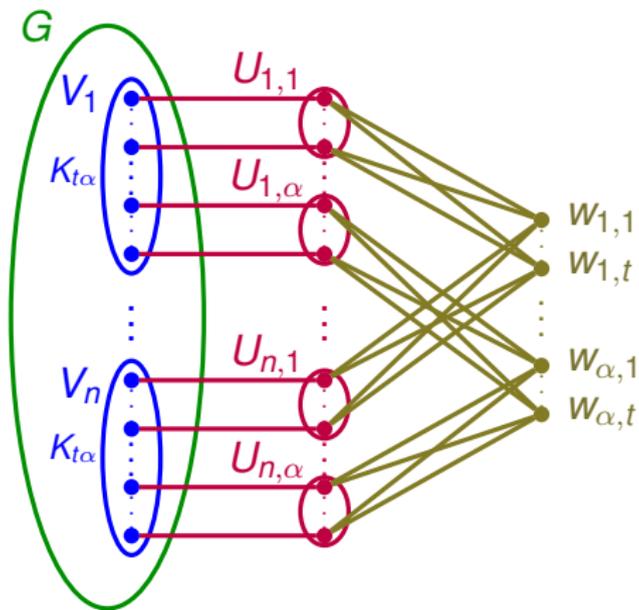


New results

Theorem

For every positive integer t , MIN- t -TOUGH is DP-complete.

α -CRITICAL \leq_p MIN- t -TOUGH



New results

Theorem

For every $b > 0$ integer and $t = 1/b$, MIN- t -TOUGH is DP-complete.

ALMOST-MIN-1-TOUGH \leq_P MIN- t -TOUGH

Theorem

For every $t = a/b \leq 1/2$ positive rational number, MIN- t -TOUGH is DP-complete.

ALMOST-MIN-1-TOUGH \leq_P MIN- t -TOUGH

Question

Is it DP-complete for other t values?

The End

