On Longest Cycles in Essentially 4 - connected Planar Graphs

J. Harant

TU Ilmenau, Germany

(joint work with I. Fabrici and S. Jendrol', Košice, Slovakia)

Gent, August 2016

Notation

- All graphs G considered here are *polyhedral*, i.e. planar and 3-connected.
- $n = n(G)$ - order of G
- $circ(G)$ - length of a longest cycle of G (*circumference* of G)
- If $circ(G) = n$, then G is *hamiltonian* and a longest cycle is a *hamiltonian cycle*.

Notation

- All graphs G considered here are *polyhedral*, i.e. planar and 3-connected.
- $n = n(G)$ - order of G
- $circ(G)$ - length of a longest cycle of G (*circumference* of G)
- If $circ(G) = n$, then G is *hamiltonian* and a longest cycle is a *hamiltonian cycle*.

Notation

- All graphs G considered here are *polyhedral*, i.e. planar and 3-connected.
- $n = n(G)$ - order of G
- $circ(G)$ - length of a longest cycle of G (*circumference* of G)
- If $circ(G) = n$, then G is *hamiltonian* and a longest cycle is a *hamiltonian cycle*.

Notation

- All graphs G considered here are *polyhedral*, i.e. planar and 3-connected.
- $n = n(G)$ - order of G
- $circ(G)$ - length of a longest cycle of G (*circumference* of G)
- If $circ(G) = n$, then G is *hamiltonian* and a longest cycle is a *hamiltonian cycle*.

Circumference $circ(G)$ of a polyhedral graph G

J.W. Moon and L. Moser, 1963

There are infinitely many maximal *planar graphs G with*

$$circ(G) \leq 9n(G)^{\log_3 2} \quad (\log_3 2 = 0.6309...).$$

- Is the exponent $\log_3 2$ smallest possible for *maximal* planar graphs ?
- Can $\log_3 2$ be decreased if *arbitrary* polyhedral graphs are considered ?
- Later the coefficient 9 was decreased several times.

Circumference $circ(G)$ of a polyhedral graph G

J.W. Moon and L. Moser, 1963

There are infinitely many maximal planar graphs G with

$$circ(G) \leq 9n(G)^{\log_3 2} \quad (\log_3 2 = 0.6309...).$$

- Is the exponent $\log_3 2$ smallest possible for *maximal* planar graphs ?
- Can $\log_3 2$ be decreased if *arbitrary* polyhedral graphs are considered ?
- Later the coefficient 9 was decreased several times.

Circumference $circ(G)$ of a polyhedral graph G

J.W. Moon and L. Moser, 1963

There are infinitely many maximal planar graphs G with

$$circ(G) \leq 9n(G)^{\log_3 2} \quad (\log_3 2 = 0.6309...).$$

- Is the exponent $\log_3 2$ smallest possible for *maximal* planar graphs ?
- Can $\log_3 2$ be decreased if *arbitrary* polyhedral graphs are considered ?
- Later the coefficient 9 was decreased several times.

Circumference $circ(G)$ of a polyhedral graph G

J.W. Moon and L. Moser, 1963

There are infinitely many maximal planar graphs G with

$$circ(G) \leq 9n(G)^{\log_3 2} \quad (\log_3 2 = 0.6309...).$$

- Is the exponent $\log_3 2$ smallest possible for *maximal* planar graphs ?
- Can $\log_3 2$ be decreased if *arbitrary* polyhedral graphs are considered ?
- Later the coefficient 9 was decreased several times.

Circumference $circ(G)$ of a polyhedral graph G

J.W. Moon and L. Moser, 1963

There are infinitely many maximal planar graphs G with

$$circ(G) \leq 9n(G)^{\log_3 2} \quad (\log_3 2 = 0.6309...).$$

G. Chen and X. Yu, 2002

There is a positive constant c such that $circ(G) \geq c \cdot n(G)^{\log_3 2}$ for an arbitrary polyhedral graph G on n vertices.

T. Tutte, 1956

A 4-connected planar graph is hamiltonian.

Circumference $circ(G)$ of a polyhedral graph G

J.W. Moon and L. Moser, 1963

There are infinitely many maximal *planar graphs G with*

$$circ(G) \leq 9n(G)^{\log_3 2} \quad (\log_3 2 = 0.6309...).$$

G. Chen and X. Yu, 2002

There is a positive constant c such that $circ(G) \geq c \cdot n(G)^{\log_3 2}$ for an arbitrary *polyhedral graph G on n vertices.*

T. Tutte, 1956

A 4-connected *planar graph is hamiltonian.*

What happens between 3-connected and 4-connected ?

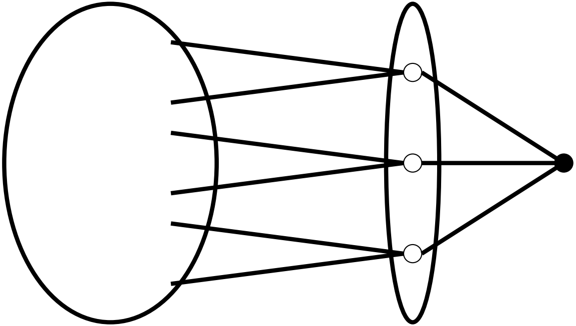
Definition

A graph G is essentially 4-connected *if G is 3-connected and each 3-separator forms the neighborhood of a vertex of degree 3.*

What happens between *3-connected* and *4-connected* ?

Definition

A graph G is essentially 4-connected *if G is* 3*-connected and each* 3*-separator forms the neighborhood of a vertex of degree* 3*.*



Let G be polyhedral and essentially 4-connected

- $circ(G) \geq \frac{2n(G)+4}{5}$.
  (Jackson, Wormald 1992)

- $circ(G) \geq \frac{3}{4}n(G)$ if G is *cubic*.
  (Grünbaum, Malkevitch 1976, Zhang 1987)

- If $c > \frac{2}{3}$, then there is an infinite family of graphs G such that $circ(G) \leq c \cdot n(G)$.
  (Jackson, Wormald 1992)

- The last statement is even true for *maximal* planar graphs.

Let G be polyhedral and essentially 4-connected

- $circ(G) \geq \frac{2n(G)+4}{5}$.
(Jackson, Wormald 1992)

- $circ(G) \geq \frac{3}{4}n(G)$ if G is *cubic*.
(Grünbaum, Malkevitch 1976, Zhang 1987)

- If $c > \frac{2}{3}$, then there is an infinite family of graphs G such that $circ(G) \leq c \cdot n(G)$.
(Jackson, Wormald 1992)

- The last statement is even true for *maximal* planar graphs.

Let G be polyhedral and essentially 4-connected

- $circ(G) \geq \frac{2n(G)+4}{5}$.
  (Jackson, Wormald 1992)
- $circ(G) \geq \frac{3}{4}n(G)$ if G is *cubic*.
  (Grünbaum, Malkevitch 1976, Zhang 1987)
- If $c > \frac{2}{3}$, then there is an infinite family of graphs G such that $circ(G) \leq c \cdot n(G)$.
  (Jackson, Wormald 1992)
- The last statement is even true for *maximal* planar graphs.

Let G be polyhedral and essentially 4-connected

- $circ(G) \geq \frac{2n(G)+4}{5}$.
  (Jackson, Wormald 1992)
- $circ(G) \geq \frac{3}{4}n(G)$ if G is *cubic*.
  (Grünbaum, Malkevitch 1976, Zhang 1987)
- If $c > \frac{2}{3}$, then there is an infinite family of graphs G such that $circ(G) \leq c \cdot n(G)$.
  (Jackson, Wormald 1992)
- The last statement is even true for *maximal* planar graphs.

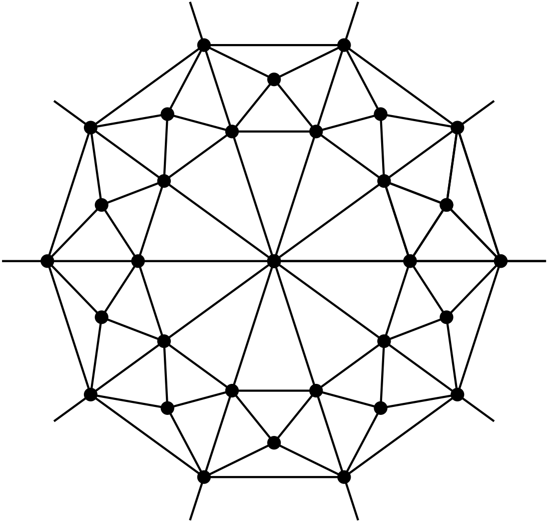
Let G be polyhedral and essentially 4-connected

- $circ(G) \geq \frac{2n(G)+4}{5}$.
  (Jackson, Wormald 1992)

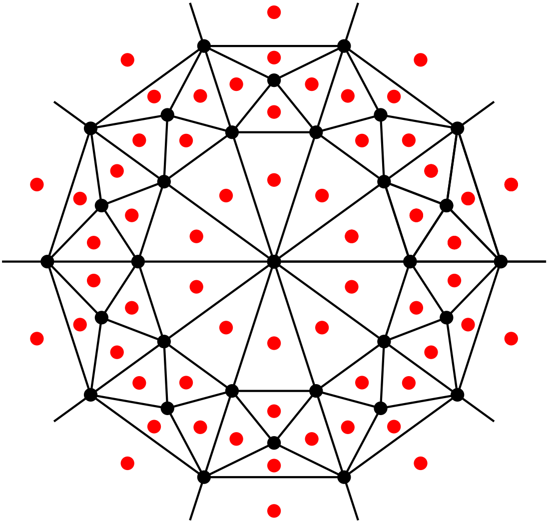
- $circ(G) \geq \frac{3}{4}n(G)$ if G is *cubic*.
  (Grünbaum, Malkevitch 1976, Zhang 1987)

- If $c > \frac{2}{3}$, then there is an infinite family of graphs G such that $circ(G) \leq c \cdot n(G)$.
  (Jackson, Wormald 1992)

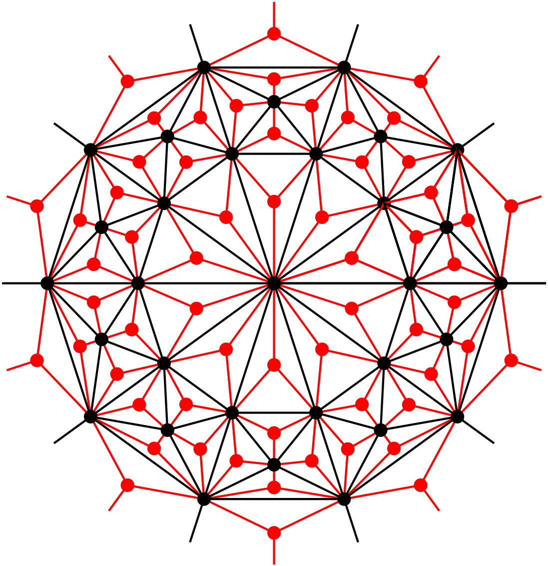
- The last statement is even true for *maximal* planar graphs.



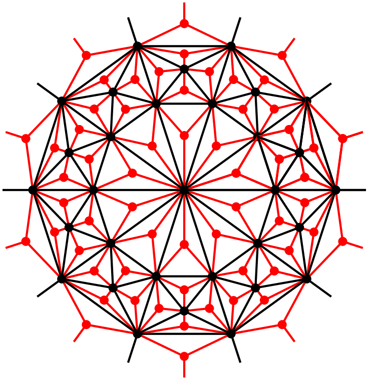
A 4-connected maximal planar graph G' on 32 vertices.



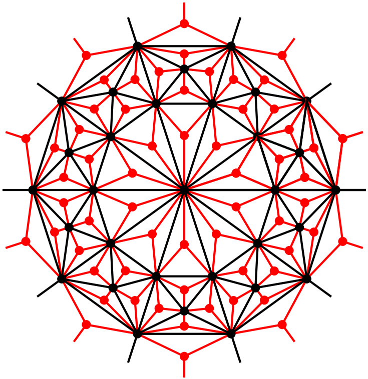
$2 \times 32 - 4 = 60$ *red vertices*



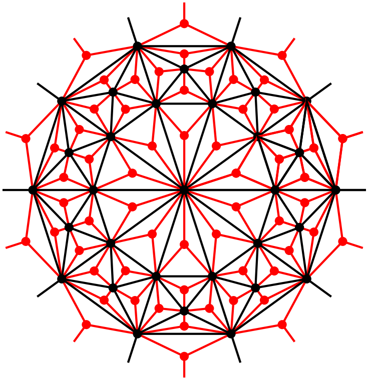
A essentially 4-connected maximal planar graph G on $32 + 2 \times 32 - 4 = 92$ vertices.



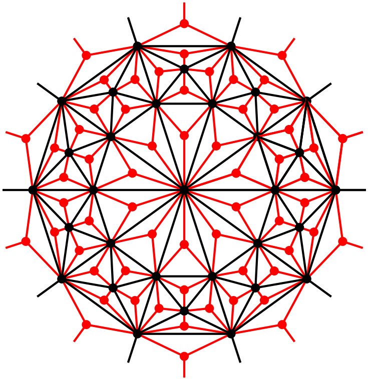
- G has 32 black vertices
- the *red vertices* are independent
- a longest cycle of G has at most $2 \times 32 = 64$ vertices
- there is a cycle on 64 vertices



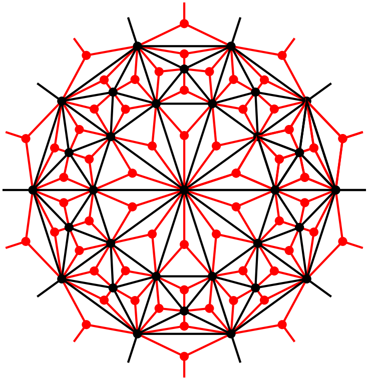
- G has 32 black vertices
- the *red vertices* are independent
- a longest cycle of G has at most $2 \times 32 = 64$ vertices
- there is a cycle on 64 vertices



- G has 32 black vertices
- the *red vertices* are independent
- a longest cycle of G has at most $2 \times 32 = 64$ vertices
- there is a cycle on 64 vertices



- G has 32 black vertices
- the *red vertices* are independent
- a longest cycle of G has at most $2 \times 32 = 64$ vertices
- there is a cycle on 64 vertices



- G has 32 black vertices
- the *red vertices* are independent
- a longest cycle of G has at most $2 \times 32 = 64$ vertices
- there is a cycle on 64 vertices

Infinitely many essentially 4-connected maximal planar graphs G with $circ(G) \leq \frac{2n(G)+8}{3}$

- G' - a *4-connected maximal plane* graph on n' vertices.
- G - obtained from G' by inserting a new vertex into each face of G' and connecting it with the tree boundary vertices of that face by an edge.
- G is an *essentially 4-connected maximal plane* graph on $n = n' + (2n' - 4)$ vertices.
- The $2n' - 4$ vertices in $V(G) \setminus V(G')$ are pairwise independent.
- Hence, each cycle of G contains at most $2n' = \frac{2n+8}{3}$ vertices.

Infinitely many essentially 4-connected maximal planar graphs G with $circ(G) \leq \frac{2n(G)+8}{3}$

- G' - a *4-connected maximal plane* graph on n' vertices.
- G - obtained from G' by inserting a new vertex into each face of G' and connecting it with the tree boundary vertices of that face by an edge.
- G is an *essentially 4-connected maximal plane* graph on $n = n' + (2n' - 4)$ vertices.
- The $2n' - 4$ vertices in $V(G) \setminus V(G')$ are pairwise independent.
- Hence, each cycle of G contains at most $2n' = \frac{2n+8}{3}$ vertices.

Infinitely many essentially 4-connected maximal planar graphs G with $circ(G) \leq \frac{2n(G)+8}{3}$

- G' - a *4-connected maximal plane* graph on n' vertices.
- G - obtained from G' by inserting a new vertex into each face of G' and connecting it with the tree boundary vertices of that face by an edge.
- G is an *essentially 4-connected maximal plane* graph on $n = n' + (2n' - 4)$ vertices.
- The $2n' - 4$ vertices in $V(G) \setminus V(G')$ are pairwise independent.
- Hence, each cycle of G contains at most $2n' = \frac{2n+8}{3}$ vertices.

Infinitely many essentially 4-connected maximal planar graphs G with $circ(G) \leq \frac{2n(G)+8}{3}$

- G' - a *4-connected maximal plane* graph on n' vertices.
- G - obtained from G' by inserting a new vertex into each face of G' and connecting it with the tree boundary vertices of that face by an edge.
- G is an *essentially 4-connected maximal plane* graph on $n = n' + (2n' - 4)$ vertices.
- The $2n' - 4$ vertices in $V(G) \setminus V(G')$ are pairwise independent.
- Hence, each cycle of G contains at most $2n' = \frac{2n+8}{3}$ vertices.

Infinitely many essentially 4-connected maximal planar graphs G with $circ(G) \leq \frac{2n(G)+8}{3}$

- G' - a *4-connected maximal plane* graph on n' vertices.
- G - obtained from G' by inserting a new vertex into each face of G' and connecting it with the tree boundary vertices of that face by an edge.
- G is an *essentially 4-connected maximal plane* graph on $n = n' + (2n' - 4)$ vertices.
- The $2n' - 4$ vertices in $V(G) \setminus V(G')$ are pairwise independent.
- Hence, each cycle of G contains at most $2n' = \frac{2n+8}{3}$ vertices.

Infinitely many essentially 4-connected maximal planar graphs G with $circ(G) \leq \frac{2n(G)+8}{3}$

- G' - a *4-connected maximal plane* graph on n' vertices.
- G - obtained from G' by inserting a new vertex into each face of G' and connecting it with the tree boundary vertices of that face by an edge.
- G is an *essentially 4-connected maximal plane* graph on $n = n' + (2n' - 4)$ vertices.
- The $2n' - 4$ vertices in $V(G) \setminus V(G')$ are pairwise independent.
- Hence, each cycle of G contains at most $2n' = \frac{2n+8}{3}$ vertices.

Let G be polyhedral and essentially 4 - connected

- $circ(G) \geq \frac{2n(G)+4}{5}$ - Jackson, Wormald 1992
  There is an infinite family of *essentially 4-connected maximal planar* graphs G such that $circ(G) \leq \frac{2n(G)+8}{3}$.
- $circ(G) \geq \frac{3}{4}n(G)$ if G is *cubic* - Grünbaum, Malkevitch 1976, Zhang 1987
  There is an infinite family of *essentially 4-connected cubic planar* graphs G such that $circ(G) \leq \frac{76}{77}n(G)$.

I. Fabrici, J.H., S. Jendrol, 2008, 2016

(i) $circ(G) \geq \frac{n(G)+4}{2}$.

(ii) $circ(G) \geq \frac{3}{5}n(G)$ *if* $\Delta = 4$.

(iii) $circ(G) \geq \frac{13}{21}(n(G)+4)$ *if* G *is* maximal *planar.*

Let G be polyhedral and essentially 4 - connected

- $circ(G) \geq \frac{2n(G)+4}{5}$ - Jackson, Wormald 1992
  There is an infinite family of *essentially 4-connected maximal planar* graphs G such that $circ(G) \leq \frac{2n(G)+8}{3}$.
- $circ(G) \geq \frac{3}{4}n(G)$ if G is *cubic* - Grünbaum, Malkevitch 1976, Zhang 1987
  There is an infinite family of *essentially 4-connected cubic planar* graphs G such that $circ(G) \leq \frac{76}{77}n(G)$.

I. Fabrici, J.H., S. Jendrol, 2008, 2016

(i) $circ(G) \geq \frac{n(G)+4}{2}$.

(ii) $circ(G) \geq \frac{3}{5}n(G)$ *if* $\Delta = 4$.

(iii) $circ(G) \geq \frac{13}{21}(n(G)+4)$ *if G is* maximal *planar.*

Let G be polyhedral and essentially 4 - connected

- $circ(G) \geq \frac{2n(G)+4}{5}$ - Jackson, Wormald 1992
  There is an infinite family of *essentially 4-connected maximal planar* graphs G such that $circ(G) \leq \frac{2n(G)+8}{3}$.
- $circ(G) \geq \frac{3}{4}n(G)$ if G is *cubic* - Grünbaum, Malkevitch 1976, Zhang 1987
  There is an infinite family of *essentially 4-connected cubic planar* graphs G such that $circ(G) \leq \frac{76}{77}n(G)$.

I. Fabrici, J.H., S. Jendrol, 2008, 2016

(i) $circ(G) \geq \frac{n(G)+4}{2}$.

(ii) $circ(G) \geq \frac{3}{5}n(G)$ *if* $\Delta = 4$.

(iii) $circ(G) \geq \frac{13}{21}(n(G)+4)$ *if G is* maximal *planar.*

Let G be polyhedral and essentially 4 - connected

- $circ(G) \geq \frac{2n(G)+4}{5}$ - Jackson, Wormald 1992
  There is an infinite family of *essentially 4-connected maximal planar* graphs G such that $circ(G) \leq \frac{2n(G)+8}{3}$.
- $circ(G) \geq \frac{3}{4}n(G)$ if G is *cubic* - Grünbaum, Malkevitch 1976, Zhang 1987
  There is an infinite family of *essentially 4-connected cubic planar* graphs G such that $circ(G) \leq \frac{76}{77}n(G)$.

I. Fabrici, J.H., S. Jendrol, 2008, 2016

(i) $circ(G) \geq \frac{n(G)+4}{2}$.

(ii) $circ(G) \geq \frac{3}{5}n(G)$ *if* $\Delta = 4$.

(iii) $circ(G) \geq \frac{13}{21}(n(G)+4)$ *if G is* maximal *planar.*

Let G be polyhedral and essentially 4 - connected

- $circ(G) \geq \frac{2n(G)+4}{5}$ - Jackson, Wormald 1992
  There is an infinite family of *essentially 4-connected maximal planar* graphs G such that $circ(G) \leq \frac{2n(G)+8}{3}$.
- $circ(G) \geq \frac{3}{4}n(G)$ if G is *cubic* - Grünbaum, Malkevitch 1976, Zhang 1987
  There is an infinite family of *essentially 4-connected cubic planar* graphs G such that $circ(G) \leq \frac{76}{77}n(G)$.

I. Fabrici, J.H., S. Jendrol, 2008, 2016

(i) $circ(G) \geq \frac{n(G)+4}{2}$.

(ii) $circ(G) \geq \frac{3}{5}n(G)$ *if* $\Delta = 4$.

(iii) $circ(G) \geq \frac{13}{21}(n(G)+4)$ *if G is maximal planar.*

Let G be polyhedral and essentially 4 - connected

- $circ(G) \geq \frac{2n(G)+4}{5}$ - Jackson, Wormald 1992
  There is an infinite family of *essentially 4-connected maximal planar* graphs G such that $circ(G) \leq \frac{2n(G)+8}{3}$.
- $circ(G) \geq \frac{3}{4}n(G)$ if G is *cubic* - Grünbaum, Malkevitch 1976, Zhang 1987
  There is an infinite family of *essentially 4-connected cubic planar* graphs G such that $circ(G) \leq \frac{76}{77}n(G)$.

I. Fabrici, J.H., S. Jendrol, 2008, 2016

(i) $circ(G) \geq \frac{n(G)+4}{2}$.

(ii) $circ(G) \geq \frac{3}{5}n(G)$ *if* $\Delta = 4$.

(iii) $circ(G) \geq \frac{13}{21}(n(G)+4)$ *if* G *is maximal planar.*

Let G be polyhedral and essentially 4 - connected

- $circ(G) \geq \frac{2n(G)+4}{5}$ - Jackson, Wormald 1992
  There is an infinite family of *essentially 4-connected maximal planar* graphs G such that $circ(G) \leq \frac{2n(G)+8}{3}$.
- $circ(G) \geq \frac{3}{4}n(G)$ if G is *cubic* - Grünbaum, Malkevitch 1976, Zhang 1987
  There is an infinite family of *essentially 4-connected cubic planar* graphs G such that $circ(G) \leq \frac{76}{77}n(G)$.

I. Fabrici, J.H., S. Jendrol, 2008, 2016

(i) $circ(G) \geq \frac{n(G)+4}{2}$.

(ii) $circ(G) \geq \frac{3}{5}n(G)$ *if* $\Delta = 4$.

(iii) $circ(G) \geq \frac{13}{21}(n(G)+4)$ *if* G *is* maximal *planar.*

Sketch of the proof of $circ(G) \geq \frac{n(G)+4}{2}$.

- A cycle C of G is an *outer-independent-3-cycle* (OI3-cycle), if $V(G) \setminus V(C)$ is an independent set of vertices and $d(x) = 3$ for all $x \in V(G) \setminus V(C)$.
- Lemma: If G is an essentially 4-connected planar graph, then G contains an OI3-cycle C.
- Let C be a longest OI3-cycle of G.
- For each edge xy of C, x and y do not have a common neighbor in $int(C) \cap V(G)$.
- Lemma: If C is a cycle of a plane graph G on at least 4 vertices such that $int(C) \cap V(G)$ is an independent set of vertices of degree 3 in G and, for each edge xy of C, x and y do not have a common neighbor in $int(C) \cap V(G)$, then $|int(C) \cap V(G)| \leq \frac{1}{2}(|V(C)| - 4)$.
- $n(G) = |V(C)| + |int(C) \cap V(G)| + |ext(C) \cap V(G)| \leq 2|V(C)| - 4$.

Sketch of the proof of $circ(G) \geq \frac{n(G)+4}{2}$.

- A cycle C of G is an *outer-independent-3-cycle* (OI3-cycle), if $V(G) \setminus V(C)$ is an independent set of vertices and $d(x) = 3$ for all $x \in V(G) \setminus V(C)$.
- Lemma: If G is an essentially 4-connected planar graph, then G contains an OI3-cycle C.
- Let C be a longest OI3-cycle of G.
- For each edge xy of C, x and y do not have a common neighbor in $int(C) \cap V(G)$.
- Lemma: If C is a cycle of a plane graph G on at least 4 vertices such that $int(C) \cap V(G)$ is an independent set of vertices of degree 3 in G and, for each edge xy of C, x and y do not have a common neighbor in $int(C) \cap V(G)$, then $|int(C) \cap V(G)| \leq \frac{1}{2}(|V(C)| - 4)$.
- $n(G) = |V(C)| + |int(C) \cap V(G)| + |ext(C) \cap V(G)| \leq 2|V(C)| - 4$.

Sketch of the proof of $circ(G) \geq \frac{n(G)+4}{2}$.

- A cycle C of G is an *outer-independent-3-cycle* (OI3-cycle), if $V(G) \setminus V(C)$ is an independent set of vertices and $d(x) = 3$ for all $x \in V(G) \setminus V(C)$.
- Lemma: If G is an essentially 4-connected planar graph, then G contains an OI3-cycle C.
- Let C be a longest OI3-cycle of G.
- For each edge xy of C, x and y do not have a common neighbor in $int(C) \cap V(G)$.
- Lemma: If C is a cycle of a plane graph G on at least 4 vertices such that $int(C) \cap V(G)$ is an independent set of vertices of degree 3 in G and, for each edge xy of C, x and y do not have a common neighbor in $int(C) \cap V(G)$, then $|int(C) \cap V(G)| \leq \frac{1}{2}(|V(C)| - 4)$.
- $n(G) = |V(C)| + |int(C) \cap V(G)| + |ext(C) \cap V(G)| \leq 2|V(C)| - 4$.

Sketch of the proof of $circ(G) \geq \frac{n(G)+4}{2}$.

- A cycle C of G is an *outer-independent-3-cycle* (OI3-cycle), if $V(G) \setminus V(C)$ is an independent set of vertices and $d(x) = 3$ for all $x \in V(G) \setminus V(C)$.
- Lemma: If G is an essentially 4-connected planar graph, then G contains an OI3-cycle C.
- Let C be a longest OI3-cycle of G.
- For each edge xy of C, x and y do not have a common neighbor in $int(C) \cap V(G)$.
- Lemma: If C is a cycle of a plane graph G on at least 4 vertices such that $int(C) \cap V(G)$ is an independent set of vertices of degree 3 in G and, for each edge xy of C, x and y do not have a common neighbor in $int(C) \cap V(G)$, then $|int(C) \cap V(G)| \leq \frac{1}{2}(|V(C)| - 4)$.
- $n(G) = |V(C)| + |int(C) \cap V(G)| + |ext(C) \cap V(G)| \leq 2|V(C)| - 4$.

Sketch of the proof of $circ(G) \geq \frac{n(G)+4}{2}$.

- A cycle C of G is an *outer-independent-3-cycle* (OI3-cycle), if $V(G) \setminus V(C)$ is an independent set of vertices and $d(x) = 3$ for all $x \in V(G) \setminus V(C)$.
- Lemma: If G is an essentially 4-connected planar graph, then G contains an OI3-cycle C.
- Let C be a longest OI3-cycle of G.
- For each edge xy of C, x and y do not have a common neighbor in $int(C) \cap V(G)$.
- Lemma: If C is a cycle of a plane graph G on at least 4 vertices such that $int(C) \cap V(G)$ is an independent set of vertices of degree 3 in G and, for each edge xy of C, x and y do not have a common neighbor in $int(C) \cap V(G)$, then $|int(C) \cap V(G)| \leq \frac{1}{2}(|V(C)| - 4)$.
- $n(G) = |V(C)| + |int(C) \cap V(G)| + |ext(C) \cap V(G)| \leq 2|V(C)| - 4$.

Sketch of the proof of $circ(G) \geq \frac{n(G)+4}{2}$.

- A cycle C of G is an *outer-independent-3-cycle* (OI3-cycle), if $V(G) \setminus V(C)$ is an independent set of vertices and $d(x) = 3$ for all $x \in V(G) \setminus V(C)$.
- Lemma: If G is an essentially 4-connected planar graph, then G contains an OI3-cycle C.
- Let C be a longest OI3-cycle of G.
- For each edge xy of C, x and y do not have a common neighbor in $int(C) \cap V(G)$.
- Lemma: If C is a cycle of a plane graph G on at least 4 vertices such that $int(C) \cap V(G)$ is an independent set of vertices of degree 3 in G and, for each edge xy of C, x and y do not have a common neighbor in $int(C) \cap V(G)$, then $|int(C) \cap V(G)| \leq \frac{1}{2}(|V(C)| - 4)$.
- $n(G) = |V(C)| + |int(C) \cap V(G)| + |ext(C) \cap V(G)| \leq 2|V(C)| - 4$.

Sketch of the proof of $circ(G) \geq \frac{n(G)+4}{2}$.

- A cycle C of G is an *outer-independent-3-cycle* (OI3-cycle), if $V(G) \setminus V(C)$ is an independent set of vertices and $d(x) = 3$ for all $x \in V(G) \setminus V(C)$.
- Lemma: If G is an essentially 4-connected planar graph, then G contains an OI3-cycle C.
- Let C be a longest OI3-cycle of G.
- For each edge xy of C, x and y do not have a common neighbor in $int(C) \cap V(G)$.
- Lemma: If C is a cycle of a plane graph G on at least 4 vertices such that $int(C) \cap V(G)$ is an independent set of vertices of degree 3 in G and, for each edge xy of C, x and y do not have a common neighbor in $int(C) \cap V(G)$, then $|int(C) \cap V(G)| \leq \frac{1}{2}(|V(C)| - 4)$.
- $n(G) = |V(C)| + |int(C) \cap V(G)| + |ext(C) \cap V(G)| \leq 2|V(C)| - 4$.

Proof of

Lemma: If C is a cycle of a plane graph G on at least 4 vertices such that $int(C) \cap V(G)$ is an independent set of vertices of degree 3 in G and, for each edge xy of C, x and y do not have a common neighbor in $int(C) \cap V(G)$, then $|int(C) \cap V(G)| \leq \frac{1}{2}(|V(C)| - 4)$.

- If $|int(C) \cap V(G)| = 0$, then nothing is to prove.
- Induction on $|V(C)|$.
- If $|V(C)| \leq 5$, then, obviously, $|int(C) \cap V(G)| = 0$.

Proof of

Lemma: If C is a cycle of a plane graph G on at least 4 vertices such that $int(C) \cap V(G)$ is an independent set of vertices of degree 3 in G and, for each edge xy of C, x and y do not have a common neighbor in $int(C) \cap V(G)$, then $|int(C) \cap V(G)| \leq \frac{1}{2}(|V(C)| - 4)$.

- If $|int(C) \cap V(G)| = 0$, then nothing is to prove.
- Induction on $|V(C)|$.
- If $|V(C)| \leq 5$, then, obviously, $|int(C) \cap V(G)| = 0$.

Proof of

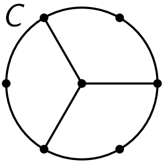
Lemma: If C is a cycle of a plane graph G on at least 4 vertices such that $int(C) \cap V(G)$ is an independent set of vertices of degree 3 in G and, for each edge xy of C, x and y do not have a common neighbor in $int(C) \cap V(G)$, then $|int(C) \cap V(G)| \leq \frac{1}{2}(|V(C)| - 4)$.

- If $|int(C) \cap V(G)| = 0$, then nothing is to prove.
- Induction on $|V(C)|$.
- If $|V(C)| \leq 5$, then, obviously, $|int(C) \cap V(G)| = 0$.

Proof of

Lemma: If C is a cycle of a plane graph G on at least 4 vertices such that $int(C) \cap V(G)$ is an independent set of vertices of degree 3 in G and, for each edge xy of C, x and y do not have a common neighbor in $int(C) \cap V(G)$, then $|int(C) \cap V(G)| \leq \frac{1}{2}(|V(C)| - 4)$.

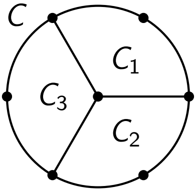
- If $|int(C) \cap V(G)| = 0$, then nothing is to prove.
- Induction on $|V(C)|$.
- If $|V(C)| \leq 5$, then, obviously, $|int(C) \cap V(G)| = 0$.



Proof of

Lemma: If C is a cycle of a plane graph G on at least 4 vertices such that $int(C) \cap V(G)$ is an independent set of vertices of degree 3 in G and, for each edge xy of C, x and y do not have a common neighbor in $int(C) \cap V(G)$, then $|int(C) \cap V(G)| \leq \frac{1}{2}(|V(C)| - 4)$.

- $|V(C)| \geq 6$ and $|int(C) \cap V(G)| > 0$.

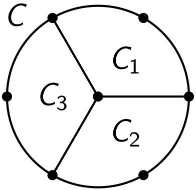


- $|int(C_i) \cap V(G)| \leq \frac{|V(C_i)|}{2} - 2$ for $i = 1, 2, 3$ (ind. hyp.).
- $|V(C_1)| + |V(C_2)| + |V(C_3)| = |V(C)| + 6$.
- $|int(C_1) \cap V(G)| + |int(C_2) \cap V(G)| + |int(C_3) \cap V(G)| = |int(C) \cap V(G)| - 1$.

Proof of

Lemma: If C is a cycle of a plane graph G on at least 4 vertices such that $int(C) \cap V(G)$ is an independent set of vertices of degree 3 in G and, for each edge xy of C, x and y do not have a common neighbor in $int(C) \cap V(G)$, then $|int(C) \cap V(G)| \leq \frac{1}{2}(|V(C)| - 4)$.

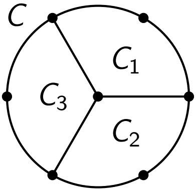
- $|V(C)| \geq 6$ and $|int(C) \cap V(G)| > 0$.



- $|int(C_i) \cap V(G)| \leq \frac{|V(C_i)|}{2} - 2$ for $i = 1, 2, 3$ (ind. hyp.).
- $|V(C_1)| + |V(C_2)| + |V(C_3)| = |V(C)| + 6$.
- $|int(C_1) \cap V(G)| + |int(C_2) \cap V(G)| + |int(C_3) \cap V(G)| = |int(C) \cap V(G)| - 1$.

Lemma: If C is a cycle of a plane graph G on at least 4 vertices such that $int(C) \cap V(G)$ is an independent set of vertices of degree 3 in G and, for each edge xy of C, x and y do not have a common neighbor in $int(C) \cap V(G)$, then $|int(C) \cap V(G)| \leq \frac{1}{2}(|V(C)| - 4)$.

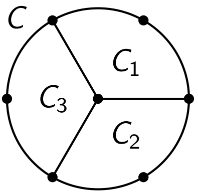
- $|V(C)| \geq 6$ and $|int(C) \cap V(G)| > 0$.



- $|int(C_i) \cap V(G)| \leq \frac{|V(C_i)|}{2} - 2$ for $i = 1, 2, 3$ (ind. hyp.).
- $|V(C_1)| + |V(C_2)| + |V(C_3)| = |V(C)| + 6$.
- $|int(C_1) \cap V(G)| + |int(C_2) \cap V(G)| + |int(C_3) \cap V(G)| = |int(C) \cap V(G)| - 1$.

Lemma: If C is a cycle of a plane graph G on at least 4 vertices such that $int(C) \cap V(G)$ is an independent set of vertices of degree 3 in G and, for each edge xy of C, x and y do not have a common neighbor in $int(C) \cap V(G)$, then $|int(C) \cap V(G)| \leq \frac{1}{2}(|V(C)| - 4)$.

- $|V(C)| \geq 6$ and $|int(C) \cap V(G)| > 0$.



- $|int(C_i) \cap V(G)| \leq \frac{|V(C_i)|}{2} - 2$ for $i = 1, 2, 3$ (ind. hyp.).
- $|V(C_1)| + |V(C_2)| + |V(C_3)| = |V(C)| + 6$.
- $|int(C_1) \cap V(G)| + |int(C_2) \cap V(G)| + |int(C_3) \cap V(G)| = |int(C) \cap V(G)| - 1$.

Thank you for your attention !